# Math Diversion Problem 871

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I take the positivist viewpoint that a physical theory is just
a mathematical model and that it is meaningless to ask
whether it corresponds to reality. All that one can ask
is that its predictions should be in agreement
with observation. — Stephen Hawking
[*The Nature of Space and Time*, p. 3–4]

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Source:   https://www.youtube.com/watch?v=-GQFljOVZ7I
Title: Zeta Function - Part 10: The Jacobi Theta Function
  Presenter:  MrYouMath
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## 1 Introduction

This is the tenth part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

## 2 The Jacobi Theta Function – Part 10

Let's just jump in a define the Jacobi Theta Function

$$\theta(x) \equiv \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} \,. \tag{1}$$

But wait! First we need the Poisson Summation Formula:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} f(y) e^{-2\pi i k y} dy \,. \tag{2}$$

Then,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi y^2 x} e^{-2\pi iky} dy \\
&= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi y^2 x - 2\pi iky} dy \\
&= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi \left[(y+ik/x)^2 - i^2 ik^2/x^2\right]} dy \\
&= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi k^2/x} e^{-\pi(y+ik/x)^2}\, dy \\
&= \sum_{k\in\mathbb{Z}} e^{-\pi k^2/x} \int_{-\infty}^{\infty} e^{-\pi(y+ik/x)^2}\, dy \\
&= \sum_{k\in\mathbb{Z}} e^{-\pi k^2/x} \int_{-\infty+ik/x}^{\infty+ik/x} e^{-\pi x z^2}\, dz \\
&= \sum_{k\in\mathbb{Z}} e^{-\pi k^2/x} \int_{-\infty}^{\infty} e^{-\pi x z^2}\, dz\,. \qquad (3)
\end{aligned}$$

The integral on the RHS is known as the Gaussian Integral:

$$\int_{-\infty}^{\infty} e^{-\pi x z^2}\, dz = \sqrt{\frac{\pi}{\pi x}} = \frac{1}{\sqrt{x}}\,. \qquad (4)$$

So, substituting this into (3) we get

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \sum_{k\in\mathbb{Z}} e^{-\pi k^2/x} \frac{1}{\sqrt{x}}\,. \qquad (5)$$

And so going back to the definition of the theta function, we get

$$\theta(x) = \frac{1}{\sqrt{x}} \theta(\frac{1}{x})\,. \qquad (6)$$

MrYouMath added these notes:

$$z = y + ik/x \implies dy = dx, \quad \Big|_{-\infty}^{\infty} \to \Big|_{-\infty+ik/x}^{\infty+ik/x}\,. \qquad (7)$$

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \sum_{k\in\mathbb{Z}} e^{-\pi k^2/x} \int_{-\infty+ik/x}^{\infty+ik/x} e^{-\pi x z^2}\, dz\,. \qquad (8)$$

And

$$\int_{-R}^{R} e^{-\pi x z^2}\, dz = \int_{-R}^{R+ik/x} e^{-\pi x z^2}\, dz + \int_{-R+ik/x}^{R+ik/x} e^{-\pi x z^2}\, dz + \int_{R+ik/x}^{R} e^{-\pi x z^2}\, dz\,, \qquad (9)$$

where the first and third terms on the RHS are small.