Math Diversion Problem 873

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This is the normal result of any idea you have in mathematics: It turns out they're either wrong or trivial or already known.
— Richard Borcherds]

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Source:  https://www.youtube.com/watch?v=K6L4Ez4ZVZc
Title: Zeta Function - Part 11
  Presenter:  MrYouMath
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1 Introduction

This is the eleventh part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 The Riemann Functional Equation I

The Riemann Function Equation looks like the following

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\,, \tag{1}$$

where we meet the gamma function again:

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt\,. \tag{2}$$

Now, if we let $s \to s/2$:

$$\Gamma\big(\frac{s}{2}\big) = \int_0^\infty t^{\frac{s}{2}-1}e^{-t}\,dt\,. \tag{3}$$

Next, let $t = \pi n^2 x$:

$$\Gamma\big(\frac{s}{2}\big) = \int_0^\infty (\pi n^2 x)^{\frac{s}{2}-1} e^{-\pi n^2 x} \pi n^2 dx$$
$$= \int_0^\infty \pi^{\frac{s}{2}} n^s x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx\,. \qquad (4)$$

So, after a little rearranging

$$\pi^{-\frac{s}{2}}\Gamma\big(\frac{s}{2}\big)\frac{1}{n^s} = \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx\,. \qquad (5)$$

And, if we going to make a connection to the zeta function, we should sum on n:

$$\pi^{-\frac{s}{2}}\Gamma\big(\frac{s}{2}\big)\sum_{n=1}^{\infty}\frac{1}{n^s} = \int_0^\infty x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx\,. \qquad (6)$$

Or

$$\pi^{-\frac{s}{2}}\Gamma\big(\frac{s}{2}\big)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx\,. \qquad (7)$$

And now we bring into this the Jacobi Theta function and thew related ψ function:

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} = 1 + 2\psi(x)\,, \qquad (8)$$

where $\psi(x) \equiv \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. Now we multiply the theta function by $x^{s/2-1}$ and integrate from zero to infinity, then the RHS of (7) takes the form

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)\,dx = \int_1^\infty x^{\frac{s}{2}-1}\psi(x)\,dx + \int_0^1 x^{\frac{s}{2}-1}\psi(x)\,dx\,, \qquad (9)$$

which is true because of the additivity of integrals. Remembering our result that

$$\theta(x) = \frac{1}{\sqrt{x}}\theta\big(\frac{1}{x}\big)\,, \qquad (10)$$

and this re-expressed in terms of ψ, yields

$$2\psi(x) + 1 = \frac{1}{\sqrt{x}}(2\psi\big(\frac{1}{x}\big) + 1)\,, \qquad (11)$$

or,

$$\psi(x) = \frac{1}{\sqrt{x}}\psi\big(\frac{1}{x}\big) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\,, \qquad (12)$$

Hence, the right-most term of the RHS of (9) becomes

$$\begin{aligned}
\int_0^1 x^{\frac{s}{2}-1}\psi(x)\,dx &= \int_0^1 x^{\frac{s}{2}-1}\Big[\frac{1}{\sqrt{x}}\psi\big(\frac{1}{x}\big) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\Big]\,dx\,, \\
&= \int_0^1 \Big[x^{\frac{s}{2}-\frac{3}{2}}\psi\big(\frac{1}{x}\big) + \frac{1}{2}\Big(x^{\frac{s}{2}-\frac{3}{2}} - x^{\frac{s}{2}-1}\Big)\Big]\,dx \\
&= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\big(\frac{1}{x}\big)\,dx + \frac{1}{2}\Big[\frac{1}{s/2-1/2}x^{\frac{s}{2}-\frac{1}{2}} - \frac{1}{s/2}x^{\frac{s}{2}}\Big]_0^1 \\
&= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\big(\frac{1}{x}\big)\,dx + \frac{1}{s(s-1)}\,, \qquad (13)
\end{aligned}$$

Now we perform a variable transformation of $x = \dfrac{1}{u}$, $dx = -\dfrac{1}{u^2}\,du$ with $\big|_0^1 \to \big|_\infty^1$:

$$\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\big(\frac{1}{x}\big)\,dx + \frac{1}{s(s-1)} \to \int_\infty^1 \Big(\frac{1}{u}\Big)^{\frac{s}{2}-\frac{3}{2}}\psi(u)\frac{-du}{u^2} + \frac{1}{s(s-1)}\,, \qquad (14)$$

And now let $u \to x$ and use the minus sign to flip the limits of integration brings (13) to:

$$\int_0^1 x^{\frac{s}{2}-1}\psi(x)\,dx = \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)\,dx + \frac{1}{s(s-1)}\,. \qquad (15)$$

Substituting this into (9), we have that

$$\begin{aligned}
\int_0^\infty x^{\frac{s}{2}-1}\psi(x)\,dx &= \int_1^\infty x^{\frac{s}{2}-1}\psi(x)\,dx + \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)\,dx + \frac{1}{s(s-1)} \\
&= \int_1^\infty \big[x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\big]\psi(x)\,dx + \frac{1}{s(s-1)}\,, \qquad (16)
\end{aligned}$$

Remember that

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\psi(x)\,dx\,, \qquad (17)$$

Therefore,

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty \big[x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}}\big]\frac{\psi(x)}{x}\,dx + \frac{1}{s(s-1)}\,, \qquad (18)$$

Next, notice that the entire RHS is invariant if we let $s \to s-1$, then

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\,. \qquad (19)$$