Math Diversion Problem 875

P. Reany

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Intelligence — the ability to learn,
understand, and apply knowledge.
— Jamie Edwards
[Healthy Developer]

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Source:  https://www.youtube.com/watch?v=TnRnlJBecRg
    https://www.youtube.com/watch?v=G-fqe3BkBnE
Title: Zeta Function - Part 12--13
  Presenter:  MrYouMath
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1 Introduction

This is the twelfth and thirteenth parts of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 From video 11

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\,. \tag{1}$$

3 The Zeta Function: Riemann Functional Eq. 2 (Part 12)

We will use the Riemann Functional Eq. (1) to derive its alternative form:

$$\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s)\,. \tag{2}$$

To accomplish this, we need two other results. To get the first one, we begin with the Legendre Duplication Formula,[1]

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma\left(s\right)\Gamma\left(s+1/2\right)\,, \tag{3}$$

and then replace s by $s/2$ and rearrange factors to get:

$$\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\,. \tag{4}$$

Next, we need to get:

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}\right) = \frac{\pi}{\cos\dfrac{\pi s}{2}}\,. \tag{5}$$

We'll accomplish this by starting with the Euler reflection formula

$$\Gamma\left(s\right)\Gamma\left(1-s\right) = \frac{\pi}{\sin\dfrac{\pi s}{2}}\,, \tag{6}$$

and then let $s \to (s+1)/2$:

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right) = \frac{\pi}{\sin\left(\dfrac{\pi s}{2}+\dfrac{\pi}{2}\right)}\,, \tag{7}$$

which simplifies to

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\cos\dfrac{\pi s}{2}}\,. \tag{8}$$

Now, multiply (1) through by $\Gamma\left(\frac{s+1}{2}\right)$:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\,. \tag{9}$$

So now we can use the Legendre duplication formula on the LHS and on the RHS we'll use (8):

$$\pi^{-s/2}\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s)\zeta(s) = \pi^{-\frac{1-s}{2}}\frac{\pi}{\cos\dfrac{\pi s}{2}}\zeta(1-s)\,. \tag{10}$$

[1]See MrYouMath: Gamma Function - Part 11 - Legendre Duplication Formula, Don't get confused! MrYouMath has two related series: one on the Riemann Zeta function and another one on the Gamma Function.

After some algebra, we have that

$$\zeta(1-s) = \frac{2}{(2\pi)^s} \cos\frac{\pi s}{2}\,\Gamma(s)\zeta(s)\,. \tag{11}$$

Next, we do the replacement: $1-s \to s$. Then,

$$\zeta(s) = \frac{2}{(2\pi)^{1-s}} \cos\frac{\pi(1-s)}{2}\,\Gamma(1-s)\zeta(1-s)\,. \tag{12}$$

On using standard trig identities as before, we have that

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s)\,, \tag{13}$$

which concludes the establishment of (2).

4 Trivial Zeros of the Zeta Function (Part 13)

It's obvious from (13) that because the sine function has zeros at integer multiples of π, that the zeta function has zeros for $s = -2k$ where $k \in \mathbb{Z}^+$. Or,

$$\zeta(-2k) = 0\,. \tag{14}$$

These zeroes are referred to as 'trivial' because it requires almost no effort to establish this fact.