Math Diversion Problem 877

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There's a game that I know some category theorists play,
which is when one person is supposed to state a theorem
and the other person is supposed to figure out how to
prove the theorem using the Yoneda lemma, and
it's sort of surprising how often that works.
— Emily Riehl, July 2020

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Source:  https://www.youtube.com/watch?v=QfDbF_qlp58
Title: Zeta Function - Part 14
  Presenter:  MrYouMath
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1 Introduction

This is the fourteenth part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 From video 11

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\,. \tag{1}$$

3 Zeta Function – Riemann Xi Function: Splitting the Difference

First, we return to the Riemann Functional Equation:

$$\pi^{-s/2}\Gamma\big(\frac{s}{2}\big)\zeta(s) = \int_1^\infty \big[x^{\frac{s}{2}} + x^{\frac{s-1}{2}}\big]\frac{\psi(x)}{x}\,dx - \frac{1}{s(1-s)}\,. \tag{2}$$

Now, multiply through by $\frac{1}{2}s(1-s)$:

$$\tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma\big(\frac{s}{2}\big)\zeta(s) = \tfrac{1}{2}s(s-1)\int_1^\infty \big[x^{\frac{s}{2}} + x^{\frac{s-1}{2}}\big]\frac{\psi(x)}{x}\,dx + \tfrac{1}{2}\,. \tag{3}$$

Define $\xi(s)$ by

$$\xi(s) \equiv \tfrac{1}{2} s(s-1)\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)\,. \tag{4}$$

From this we get that

$$\xi(s) = \xi(1-s)\,. \tag{5}$$

Now, let

$$s = \tfrac{1}{2} + it\,, \quad \text{where} \quad t \in \mathbb{C}\,. \tag{6}$$

Then

$$\xi(\tfrac{1}{2} + it) = \xi(\tfrac{1}{2} - it)\,. \tag{7}$$