Math Diversion Problem 878

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I love it when a plan comes together.
— Hannibal Smith, *The A-Team*

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Source: The Ether of Great Mathematical Ideas
Title: A Lambert Function problem.
 Presenter: Patrick
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1 Problem

Given the relation

$$\phi = \sqrt{3}^{\sqrt{3}^{\sqrt{3}^{\cdot^{\cdot^{\cdot}}}}}\,, \tag{1}$$

find the value of ϕ in some finite form.

2 Solution

Noting that ϕ has a self-referential subpart, we can write

$$\phi = \sqrt{3}^{\phi}\,. \tag{2}$$

And now we have something that looks like a job for the Lambert W function.[1]

After a little algebra, we can write

$$\phi\sqrt{3}^{-\phi} = 1\,. \tag{3}$$

After a bit more algebra, we have that

$$-\phi\sqrt{3}^{-\phi} = -1\,. \tag{4}$$

Now we're in a position to apply the Lambert W function across this last equation, to get

$$-\phi = W_{\sqrt{3}}(-1) = \frac{W_n(-1 \cdot \ln\sqrt{3})}{\ln\sqrt{3}}\,, \tag{5}$$

[1] See the Appendix.

where $n \in \mathbb{Z}$.

Finally,

$$\phi = -2\frac{W_n(-\frac{1}{2}\ln 3)}{\ln 3}\,, \tag{6}$$

which WolframAlpha also gets.

3 Appendix: Lambert W Function

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{7}$$

then

$$z = W(B)\,, \tag{8}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y\ln y = B\,, \tag{9}$$

then

$$\ln y = W(y\ln y) = W(B)\,. \tag{10}$$

The following is the 'Lambert W function base s'[2], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{11}$$

which looks very similar to (7). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{12}$$

If s is an integer, so as not to confuse s with the integer subscripts n, I usually write

$$x = W_{(s)}(xs^x) \equiv \frac{W_n(A\ln s)}{\ln s}\,. \tag{13}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W_n(A\ln e)}{\ln e} = W_n(A)\,, \tag{14}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

[2]This notation I invented myself.

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma) e^{W_n(\gamma)} \,. \tag{15}$$