

Math Diversion Problem 894

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Easy to criticize, more difficult to be correct.
— Charlie Chan

Source: <https://www.youtube.com/watch?v=q2Xydd0gPjo&t=2s>
Title: The Seemingly Impossible Equation That Has a
Beautiful Solution
Presenter: Mental Math

1 Problem

Given the relation

$$\pi^x = x^\pi \tag{1}$$

solve for real values of x .

By inspection, we can see that one solution is $x = \pi$. Just the same, there are two real solutions and we'll find them both by the same method, which I'd say looks like a job for the Lambert W function!

2 Solution

We'll begin by taking the $1/\pi$ root on both sides of (1), to get

$$(\pi^x)^{1/\pi} = x \tag{2}$$

or

$$(\pi^{1/\pi})^x = x. \tag{3}$$

For simplification, let us define

$$\beta \equiv \pi^{1/\pi} = 1.4396, \tag{4}$$

where I used a calculator, so then

$$\beta^x = x. \tag{5}$$

Next, we employ a little algebra to get

$$1 = x\beta^{-x}. \tag{6}$$

Then we multiply through by -1 to put the equation in the appropriate form.

$$-1 = -x\beta^{-x}. \quad (7)$$

Now we flip sides and the apply Lambert W function base β : ¹

$$-x = W_{\beta}(-1). \quad (8)$$

Next, we multiply through by -1 and use one of the lemma in the Appendix, we get

$$x = -\frac{W_n(-1 \cdot \ln \beta)}{\ln \beta} = -\frac{W_n(-0.364)}{0.364}, \quad (9)$$

where n is an integer. The two real values occur for $n = -1$ and for $n = 0$.

I used WolframAlpha to calculate this Lambert W_n function for me, using the separate commands

`ProductLog[-1,-0.364]` and also `ProductLog[0,-0.364]`.

$$\text{ProductLog}[n, -0.364] = \begin{cases} -1.15276 & \text{for } n = -1, \\ -0.8614 & \text{for } n = 0. \end{cases} \quad (10)$$

On substituting these values into (9), we have that

$$x = \begin{cases} 3.1676 & \text{for } n = -1, \\ 2.366 & \text{for } n = 0. \end{cases} \quad (11)$$

WolframAlpha calculated the values to get

$$x = \begin{cases} 3.14159 & \text{for } n = -1, \\ 2.38218 & \text{for } n = 0. \end{cases} \quad (12)$$

So we found the approximate value for π , which is the trivial solution, and we also get the value 2.38218, which is the non-trivial solution.

3 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B, \quad (13)$$

then

$$z = W(B), \quad (14)$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

¹For explanations about what I'm doing, see the Appendix or see my short write up in PDF on the Lambert W function.

A lemma I'll need from the theory of the Lambert W function is the following:
If

$$y \ln y = B, \tag{15}$$

then

$$\ln y = W(y \ln y) = W(B). \tag{16}$$

The following is the 'Lambert W function base s '², or W_s , where s is a positive real number. Let's begin with the relation

$$xs^x = A, \tag{17}$$

which looks very similar to (13). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}. \tag{18}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A), \tag{19}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n -series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}. \tag{20}$$

²This notation I invented myself.