Math Diversion Problem 904

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The Lambert W function is a fascinating and powerful tool in
mathematics, often overlooked but invaluable for solving
complex equations and optimization problems.
— Slavisa Velickovic

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Source:  The Ether of Great Mathematical Ideas
Title: A Lambert Lemma
 Presenter: Patrick
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1 The Problem

Lemma:

Show that the solution to

$$\log x = x\gamma\,, \tag{1}$$

where the logarithm given is base 10 and γ is a real number, is

$$x = e^{-W_n(-\gamma\,\ln 10)}\,. \tag{2}$$

2 The Proof

Let's begin by rewriting the Given equation into the form

$$\frac{\log_e x}{\log_e 10} = x\gamma\,. \tag{3}$$

By noting that $\log_e = \ln$ and defining

$$\gamma' \equiv \gamma\,\ln 10\,, \tag{4}$$

(1) becomes

$$\ln x = x\gamma'\,. \tag{5}$$

Since $x \neq 0$, we can divide by x:

$$x^{-1}\ln x = \gamma'\,. \tag{6}$$

Next, we multiply through by -1 and apply some logarithmic alchemy to get

$$x^{-1}\ln x^{-1} = -\gamma' \,. \tag{7}$$

And now the big step of applying the Lambert W function across this equation, we have that

$$\ln x^{-1} = W_n(-\gamma') \text{ where } n \in \mathbb{Z} \,. \tag{8}$$

On exponentiating,

$$x^{-1} = e^{W_n(-\gamma')} \,. \tag{9}$$

And a bit of algebra along with (4), gives us

$$x = e^{-W_n(-\gamma\ \ln 10)} \,. \tag{10}$$

According to WolframAlpha, the only appropriate values for n are 0 and -1. And I leave it to the Lambert W function experts to decide on this.

3 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{11}$$

then

$$z = W(B) \,, \tag{12}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B \,, \tag{13}$$

then

$$\ln y = W(y \ln y) = W(B) \,. \tag{14}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A \,, \tag{15}$$

which looks very similar to (11). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s} \,. \tag{16}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A) \,, \tag{17}$$

[1]This notation I invented myself.

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{18}$$