Math Diversion Problem 906

P. Reany

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With me, everything turns into mathematics.
— Rene Descartes
(P.S. I calculate; therefore I am.)

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Source:  https://infinitemathworld.com/exploring-the-lambert
              -w-function-applications-and-examples/
Title: Exploring The Lambert W Function
 Presenter: Slavisa Velickovic
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1 The Problem

Maximize $\phi = x^y$ with the constraints $x + y = 4$ and $x > 0$.

2 Solution

The two givens can be combined to one equation in x:

$$\phi(x) = x^{4-x}\,. \tag{1}$$

To find the x's where ϕ takes a maximum, we need to use the constraint $\phi'(x) = 0$. This is one of those times we should employ the logarithmic derivative. So, we take the natural logarithm across (1), to get

$$\ln \phi = (4 - x)\ln x\,. \tag{2}$$

Now differentiate.

$$\frac{\phi'(x)}{\phi} = -\ln x + (4 - x)\frac{1}{x}\,. \tag{3}$$

But since we're setting $\phi'(x) = 0$, we have that

$$-\ln x + (4 - x)\frac{1}{x} = 0\,, \tag{4}$$

which we can rewrite as

$$x(\ln x + 1) = 4\,. \tag{5}$$

And this is where I got stuck.

Of all these many problems which use the Lambert W function that I've solved on this forum over the years, I've used mostly just two versions of the Lambert W to complete the solution. The first is the 'Zig' version:

$$W(xe^x) = x\,. \tag{6}$$

The second is the 'Zag' version:

$$W(x \ln x) = \ln x\,. \tag{7}$$

And the 'Zeek' versions would be all the other lemmas of the Lambert, which one can find listed on the Wikipedia page. Now, I have been so successful by using the Zag version, that I tried it at this point and failed (though that doesn't imply that it couldn't succeed). So, I did something I don't usually do, which was to look at the author's solution, and that used the Zig approach to the solution. Hence, I concluded that I had zagged when I should have zigged. A lesson to keep in mind.

So at this point, let's introduce a new variable that will facilitate us zigging.

$$y \equiv \ln x\,. \tag{8}$$

Then substitute this into (5) and massage it with a bit of algebra, to get

$$(y+1)e^y = 4\,. \tag{9}$$

So, remember that I said it's time to use the Zig form of Lambert? All we need to do now is to put (9) into the appropriate form. Obviously, then, we need to multiply across by e:

$$(y+1)e^{y+1} = 4e\,. \tag{10}$$

Now we apply the Lambert W function across this, to get

$$y + 1 = W_n(4e)\,. \tag{11}$$

According to WolframAlpha, $W_0(4e) \approx 1.79904$. Continuing with this value for n, that means[1] that

$$y \approx 0.79904\,. \tag{12}$$

Hence

$$x \approx e^{0.79904} \approx 2.223\,. \tag{13}$$

When WolframAlpha solved $x(\ln x + 1) = 4$ itself for x, it got

$$x \approx 2.223\,. \tag{14}$$

[1]The Lambert W function in WolframAlpha is accessed by the ProductLog[] command.

3 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{15}$$

then

$$z = W(B)\,, \tag{16}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{17}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{18}$$

The following is the 'Lambert W function base s'[2], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{19}$$

which looks very similar to (15). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{20}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{21}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{22}$$

[2] This notation I invented myself.