Math Diversion Problem 922

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Physics intuition completely justifies such a
transgression against mathematical rigor.
— Ettore Majorana

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Source: The Ether of Great Mathematical Ideas
Title: A Lambert W Problem
 Presenter: Patrick
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1 The Problem

Given the relation

$$x^x = 18\,, \tag{1}$$

solve for real values of x.

2 Solution

This looks like a job for the Lambert W function. But first, we take a natural logarithm.

$$x \ln x = \ln 18\,. \tag{2}$$

Next we apply the Lambert W function, to get:

$$\ln x = W_n(\ln 18) \quad n \in \mathbb{Z}\,, \tag{3}$$

which then gives us

$$x = e^{W_n(\ln 18)}\,. \tag{4}$$

According to WolframAlpha, the only real solution occurs when $n = 0$, yielding

$$x = e^{W_0(\ln 18)} = e^{W(\ln 18)}\,, \tag{5}$$

3 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{6}$$

then

$$z = W(B)\,, \tag{7}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{8}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{9}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{10}$$

which looks very similar to (6). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{11}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{12}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma) e^{W_n(\gamma)}\,. \tag{13}$$

[1] This notation I invented myself.