Math Diversion 932

P. Reany

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Chancellor, all presidents are faced with difficult decisions.
It is by their *decisions* that they are judged.
— Cardinal Barusa
[Doctor Who, *Deadly Assassin*]

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Source: https://www.youtube.com/watch?v=hzy3yUNMPuc
Title: This Nested Radical Looks Impossible
 Presenter: Mental Math
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1 The Problem

Given the relation

$$\phi = (2+\sqrt{5})^{1/3} + (2-\sqrt{5})^{1/3}\,, \tag{1}$$

simplify ϕ in terms of real values.

2 Solution

I'm going to introduce an additional variable to simplify the complexity of our equations

$$\lambda \equiv 2+\sqrt{5}\,. \tag{2}$$

Then

$$\lambda^{-1} = -(2-\sqrt{5})\,, \tag{3}$$

where the proof of this is left to the reader. Hence

$$2-\sqrt{5} = -\lambda^{-1}\,, \tag{4}$$

A couple standard results should come in handy:

$$\lambda - \lambda^{-1} = 4\,, \tag{5}$$

$$\lambda^{1/3}\lambda^{-1/3} = 1\,. \tag{6}$$

Returning to (1), we now have that

$$\phi = \lambda^{1/3} - \lambda^{-1/3}\,. \tag{7}$$

On cubing ϕ, we get

$$\begin{aligned}\phi^3 &= \lambda - 3\lambda^{2/3}\lambda^{-1/3} + 3\lambda^{1/3}\lambda^{-2/3} - \lambda^{-1} \\ &= (\lambda - \lambda^{-1}) - 3\lambda^{1/3}\lambda^{-1/3}(\lambda^{1/3} - \lambda^{-1/3}) \\ &= 4 - 3\phi\,. \end{aligned} \tag{8}$$

From this we get the cubic equation

$$\phi^3 + 3\phi - 4 = 0\,. \tag{9}$$

This is the point we should at least try to find a value for ϕ by guessing; and if we're going to start guessing, we should start simple. So, what about $\phi = 1$? Actually, that works. And that makes our first real solution (simplification) for ϕ.

Then, using long division of polynomials, we get the initial factorization of (9) to be

$$(\phi - 1)(\phi^2 + \phi + 4) = 0\,. \tag{10}$$

However, the solutions to

$$\phi^2 + \phi + 4 = 0 \tag{11}$$

are nonreal, and the problem does not require them. Therefore, the sole real solution, and thus simplification, is

$$\phi = 1\,. \tag{12}$$