Math Diversion 940

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More than a century since its debut, representation theory
has served as a key ingredient in many of the most important
discoveries in mathematics. Yet its usefulness
is still hard to perceive at first.
— Kevin Hartnett

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Source: https://www.youtube.com/watch?v=u0P7uk4KdE0
Title: Can You Solve This Impossible Exponential Equation?
 Presenter: Mental Math
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1 The Problem

Given the relation

$$e^{\sqrt{x}} = x^{\sqrt{e}}\,, \tag{1}$$

find the real, positive values of x.

This looks like a job for the Lambert W function!

Hint: Remember to look for a trivial solution.

2 Solution

One way, of many ways, to attack this problem, is to take the square root of both sides of (1):

$$\sqrt{e}^{\sqrt{x}} = \sqrt{x}^{\sqrt{e}}\,. \tag{2}$$

So, if it wasn't obvious from (1), it should be obvious at this point that the trivial solution to (1) is

$$x = e\,. \tag{3}$$

However, this solution will also appear when we attack this problem from the methods of the Lambert W function.

Now, we introduce the parameter

$$\beta \equiv \sqrt{e}^{\,1/\sqrt{e}}\,, \tag{4a}$$

and for later use,

$$\ln \beta = \frac{1}{2\sqrt{e}}\,. \tag{4b}$$

The rationale for the former of these will be immediately clear, while that for the latter will be apparent soon.

So, we take the $\sqrt{e}\,$th root of (2), to get

$$\left(\sqrt{e}^{\,1/\sqrt{e}}\right)^{\sqrt{x}} = \sqrt{x}\,, \tag{5a}$$

or preferably we use β,

$$\beta^{\sqrt{x}} = \sqrt{x}\,. \tag{5b}$$

Next, we apply simple algebra to setup the relation so that we can apply a Lambert Lemma:

$$1 = \sqrt{x}\beta^{-\sqrt{x}}\,. \tag{6}$$

One more easy step:

$$-\sqrt{x}\beta^{-\sqrt{x}} = -1\,. \tag{7}$$

Now we apply the Lambert W function, base β, to get

$$-\sqrt{x} = W_{(\beta)}(-1) = \frac{W_n(-1\cdot\ln\beta)}{\ln\beta}\,, \tag{8}$$

where n is an integer, and since we are looking for real values of x, we'll look to $n = 0, -1$ for those.

So, upon applying (4b), we have that

$$-\sqrt{x} = \frac{W_n\big(-1\cdot\frac{1}{2\sqrt{e}}\big)}{\frac{1}{2\sqrt{e}}} = 2\sqrt{e}W_n\big(-1/2\sqrt{e}\big)\,, \tag{9}$$

On squaring both sides, we get

$$x = 4e\,[\,W_n\big(-1/2\sqrt{e}\big)\,]^2\,. \tag{10}$$

At this point, I will rely heavily on WolframAlpha for help. So, to get the value for the Lambert W function for $n = 0$, I put into WolframAlpha:

$$\text{ProductLog}[0, -1/2\sqrt{e}\,] = -\frac{1}{2}\,. \tag{11}$$

Then, substituting this value into (10), we have that

$$x_0 = 4e\,[-\tfrac{1}{2}\,]^2 = e\,, \tag{12}$$

and we get our trivial solution.

Lastly, to get the value for the Lambert W function for $n = -1$, I put into WolframAlpha:

$$\text{ProductLog}[-1, -1/2\sqrt{e}\,] \approx -1.75643\,. \tag{13}$$

Then, substituting this value into (10) with $n = -1$, we have that

$$x_{-1} \approx 33.544\,, \tag{14}$$

which is close to what WolframAlpha got for solving the original equation (1), apparently by numerical methods.

3 Afterword:

I think that I had been a bit lazy. It took me only a moment to realize, using the definition of the Lambert W function,

$$W_0(ye^y) = y\,, \tag{15}$$

that for $y = -1/2$,

$$W_0(-\tfrac{1}{2}e^{-\frac{1}{2}}) = W_0\big(-\frac{1}{2\sqrt{e}}\big) = -\tfrac{1}{2}\,. \tag{16}$$

On the other hand, the help I got from WolframAlpha encouraged me to believe that an easy demonstration of (16) existed in the first place.

4 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{17}$$

then

$$z = W(B)\,, \tag{18}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{19}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{20}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{21}$$

[1]This notation I invented myself.

which looks very similar to (17). Then

$$x = W_{(s)}(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,, \tag{22}$$

where the subscript parameter (in this case s) has been put inside parentheses to indicate that the parameter is not the usual integer value, which will come later.

But when $s = e$, we have that

$$x = W_{(e)}(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{23}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{24}$$