Math Diversion Problem 953

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Good tools shorten labor.
— Charlie Chan

Source: https://www.youtube.com/watch?v=wJ8OvYj6MUg
Title: Harvard Entrance Exam Math Question
Presenter: Math Beast

1 Problem

Given the relation

$$x^{\log 2x} = 5\,, \tag{1}$$

solve for x. [Assumption: This logarithm is base 10.]

Note: I found a solution, but there might be a shorter one.

2 Solution

Start by taking the logarithm across (1).

$$\log 2x \log x = \log 5\,. \tag{2}$$

Now, add $\log 2$ to both sides (to simplify the RHS):

$$\log 2x \log x + \log 2 = \log 5 + \log 2 = \log 10 = 1\,. \tag{3}$$

Now, since $\log 2x = \log 2 + \log x$, this last equation can be expressed as

$$(\log 2 + \log x) \log x + \log 2 = 1\,. \tag{4}$$

We can clean this up by letting $\alpha \equiv \log 2$ and $y \equiv \log x$. Then, with some algebra, (4) becomes

$$y^2 + \alpha y + \alpha - 1 = 0\,. \tag{5}$$

And the roots to this are

$$
\begin{aligned}
y &= \frac{-\alpha \pm \sqrt{\alpha^2 - 4(1)(\alpha - 1)}}{2} \\
&= \frac{-\alpha \pm \sqrt{(\alpha - 2)^2}}{2} \\
&= \frac{-\alpha \pm (\alpha - 2)}{2} \\
&= \begin{cases} \dfrac{-2}{2} = -1\,, \\ \dfrac{-2\alpha + 2}{2} = 1 - \alpha\,. \end{cases} \quad .
\end{aligned}
\tag{6}
$$

Returning to $y = \log x$ then $x = 10^y$. Thus,

$$
x = \begin{cases} 10^{-1}\,, \\ 10^{1-\alpha} = 10^1 \times 10^{-\log 2} = 10 \times 10^{\log_{10} 2^{-1}} = \frac{1}{2} \times 10\,. \end{cases} \quad . \tag{7}
$$

Finally, we end up with the two solutions

$$
x = 1/10 \quad \text{and} \quad 5\,. \tag{8}
$$