Math Diversion 978

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Complex numbers play a crucial role in various fields:
Solving Algebraic Equations, Fourier Analysis, Complex
Functions, Fractals, Fast Fourier Transform, Quantum
Mechanics, Electromagnetic Theory, Relativity,
Signal Processing, Control Systems,
Electrical Engineering.
— Copilot (a shortened listing)

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Source: https://www.youtube.com/watch?v=ykR7psuKHw4
Title: GRE Mathematics Subject Test
     - Trace of self-inverse matrix
 Presenter: Math Out Loud
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1 Problem

Let A be an element of the space of 2×2 matrices over the complex numbers. Let I be the identity matrix in this same space. If $A \neq \pm I$, and $A = A^{-1}$, what is the trace of A?

2 Preliminaries

We'll assume that A has the generic form of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{1}$$

The trace of A, $\mathrm{Tr}(A)$, is the sum of the elements on its main diagonal, hence

$$\mathrm{Tr}(A) = a + d \,. \tag{2}$$

The inverse of A is

$$A^{-1} = \frac{1}{D} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} , \tag{3}$$

where D is the determinant of A, which is given by

$$D = \det(A) = ad - bc \,. \tag{4}$$

But we know that D is nonzero because otherwise A would not have an inverse.

Lastly, we note that the given constraint on A (i.e., $A = A^{-1}$) is equivalent to

$$A^2 = I \,. \tag{5}$$

Definition: Any $n \times n$ matrix whose trace is zero is said to be *traceless*.

Definition: The A we are given, when treated as an operator (on vectors), is said to be *involutory* because when applied twice in succession the net result is the identity operation.

3 Solution

So all we have to do to comply with the given information is to set the matrices in Equations (1) and (3) equal:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{D} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \,. \tag{6}$$

On equating these two matrices component-wise, we get

$$a = D^{-1} d \,, \tag{7a}$$
$$b = D^{-1}(-b) \,, \tag{7b}$$
$$c = D^{-1}(-c) \,, \tag{7c}$$
$$d = D^{-1} a \,. \tag{7d}$$

So, from Equations (7b) and (7c) we conclude either that b and c are identically zero or that $D^{-1} = -1$. So we go to cases:

Case I: $b = c = 0$. Then from (7a) and (7d) , we get

$$a = D^{-1} d \,, \tag{8a}$$
$$d = D^{-1} a \,. \tag{8b}$$

These can be simplified to

$$a = D^2 a \,, \tag{9a}$$
$$d = D^2 d \,. \tag{9b}$$

Obviously, $D = \pm 1$, and thus $D^{-1} = \pm 1$, which would give us these cases for A:

$$A = \pm \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \,. \tag{10}$$

But the determinant of A provides another constraint on a and d, namely from (4),

$$ad = \pm 1\,. \tag{11}$$

With this information then we have four possible subcases:

$$A = \pm\begin{pmatrix} a & 0 \\ 0 & \pm a^{-1} \end{pmatrix}\,. \tag{12}$$

Subcase 1:

$$A_{+,+} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\,. \tag{13}$$

But to satisfy (5),

$$A^2_{+,+} = I\,, \tag{14}$$

then $a = \pm 1$. On $a = 1$, $A_{+,+} = I$, which is disallowed. On $a = -1$,

$$A_{+,+} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I\,, \tag{15}$$

which is also disallowed.

Subcase 2:

$$A_{+,-} = \begin{pmatrix} a & 0 \\ 0 & -a^{-1} \end{pmatrix}\,. \tag{16}$$

To satisfy $A^2_{+,-} = I$, again $a = \pm 1$. But this time we get

$$A_{+,-} = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{17}$$

Both of these matrices are allowed and they're both traceless.

Subcases 3 & 4:

I won't bore the reader with similar computations. For $A_{-,-}$, we get $A_{-,-} = \pm I$, and for $A_{-,+}$, we get the same answer as in (17).

Case II: $D^{-1} = -1$ (or rather $D = -1$). So we have that

$$a = -d\,, \tag{18a}$$
$$d = -a\,. \tag{18b}$$

Thus we get

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\,. \tag{19}$$

We can ignore the values of b and c since they don't lie on the main diagonal. Anyway, obviously we get that

$$\mathrm{Tr}(A) = a + (-a) = 0\,. \tag{20}$$

So what have we determined? That all the cases in which A is nether $\pm I$ (the disallowed cases), have trace zero.

4 Comment on the Pauli matrices

This is an important result in the Pauli theory. I won't give details but the interested reader can look up the Pauli matrices. The space of 2×2 matrices over the complex numbers forms a four-dimensional vector space with complex scalars. In Pauli's arrangement of its four basis vectors, he chose I as one of them. I, of course, is not traceless, and for clarity, the problem eliminated that as a possible outcome.

All the Pauli matrices satisfy the relation $A = A^{-1}$. Hence the three Pauli matrices (which do not include the identity) are traceless, which follows from the results we just proved.

Two of the Pauli matrices (namely, σ_1, σ_2) have only zeros as their main diagonal components, so of course they are traceless. But there is also a Pauli matrix of the form

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{21}$$

which also is traceless.

The three Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{22}$$