Math Diversion 1006

P. Reany

January 9, 2026

Mathematics compares the most diverse phenomena and
discovers the secret analogies that unite them.
— Joseph Fourier

```
Source: https://www.youtube.com/watch?v=iIAxAYeidIY
Title: This Log Equation Stumped Me
 Presenter: NonsoMaths
```

1 Problem

Given the relation

$$x \log(x+1) + \log(x+1) = 2x\,, \tag{1}$$

find real solutions for x.

2 Solution

We'll begin by employing some lemmas of logarithms (see my writeup on logarithms if you want). Then we'll use some lemmas from Lambert W function theory.

First, we combine the two terms on the LHS to get,

$$\log[(x+1)^x(x+1)] = 2x\,. \tag{2}$$

Next, we'll raise 10 to the power of the last equation (just jargon), giving us

$$(x+1)^x(x+1) = 10^{2x}\,. \tag{3}$$

With just a little algebra, this becomes

$$(x+1)^{x+1} = 10^{2x}\,. \tag{4}$$

Looks to me that we can simplify by making a change of variable, like

$$u \equiv x+1\,. \tag{5}$$

Using this u in (4), we have that

$$u^u = 10^{2(u-1)}\,. \tag{6}$$

Looks to me that we can simplify again by making an alpha transformation, like

$$u \equiv 10^\alpha\,. \tag{7}$$

With this, (6) becomes

$$10^{\alpha 10^\alpha} = 10^{2(10^\alpha - 1)}\,. \tag{8}$$

On equating the exponents, we get

$$\alpha 10^\alpha = 2(10^\alpha - 1)\,, \tag{9}$$

or

$$(\alpha - 2)10^\alpha = -2\,. \tag{10}$$

Next, we make the exponent of '10' equal to the factor in front of it, so we can use a handy lemma:

$$(\alpha - 2)10^{\alpha - 2} = -2 \times 10^{-2} = -\tfrac{1}{50}\,. \tag{11}$$

Now we apply that lemma, taking the Lambert W function base 10 across the equation:

$$\alpha - 2 = W_{(10)}(-\tfrac{1}{50}) = \frac{W_n(-\frac{1}{50}\ln 10)}{\ln 10}\,, \tag{12}$$

where n is yet to be determined for real value of x. Thus

$$\alpha = \frac{W_n(-\frac{1}{50}\ln 10)}{\ln 10} + 2\,. \tag{13}$$

Hence,

$$u = 10^{\frac{W_n(-\frac{1}{50}\ln 10)}{\ln 10} + 2} = 100 \times 10^{\frac{W_n(-\frac{1}{50}\ln 10)}{\ln 10}}\,, \tag{14}$$

which can be rewritten as

$$u = 100 \times e^{W_n(-\frac{1}{50}\log 10)}\,. \tag{15}$$

So that for x we get

$$x = 100 \times e^{W_n(-\frac{1}{50}\log 10)} - 1\,. \tag{16}$$

According to WolframAlpha, the values of n that give real values for x are $n = 0, -1$.

Note:

$$10^{y/\ln 10} = e^{\ln\left(10^{y/\ln 10}\right)} = e^{(y/\ln 10)(\ln 10)} = e^y\,. \tag{17}$$