Math Diversion 1007

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> I love it when a plan comes together.
> — Hannibal Smith, *The A-Team*

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Source: https://www.youtube.com/watch?v=H1_yk1DSac8
Title: The last 2 digits of a perfect square cannot be both odd.
 Presenter: Prime Newtons
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1 Problem

Let x be a positive integer. Show the rightmost 2 digits of x^2 cannot both be odd.

2 Preparation

Let's invent some notation for analyzing digits of positive integers base ten. We really only need two. Let $r_n(y)$ be the n rightmost digits of positive integer y. Second, let $p_n(y)$ be the nth digit of y.

3 Solution

So, we are to show that if x is a positive integer then it is not possible that both $p_1(x^2)$ and $p_2(x^2)$ are odd digits, where a digit is defined as a number from 0 to 9, inclusive. Let's just do it. Obviously $0^2 = 00,\ 1^2 = 01,\ 2^2 = 04,\ 3^2 = 09$, so they don't work because zero is an even digit. Then

$$4^2 = 16,\ 5^2 = 25,\ 6^2 = 36,\ 7^2 = 49,\ 8^2 = 64,\ 9^2 = 81\,, \tag{1}$$

and each of those fail as well.[1] Hence, no single-digit positive integer will square to a two-digit number, both digits being odd.

Let

$$\alpha \equiv p_2(x) \quad \text{and} \quad \beta \equiv p_1(x)\,. \tag{2}$$

[1] This is called *exhaustive search*, by the way.

Hence,

$$r_2(x) = \alpha\beta\,, \tag{3}$$

where we see that $\alpha\beta$ is just the rightmost two digits of x, not a product.

I'm going to let the reader justify this next claim for him or herself:

$$r_2(x^2) = r_2((\alpha\beta)^2)\,. \tag{4}$$

Hint: When x has exactly two digits:

$$(\alpha\beta)^2 = (\alpha \times 10 + \beta)^2 = \alpha^2 \times 100 + 2\alpha\beta \times 10 + \beta^2\,. \tag{5}$$

Then

$$r_2((\alpha\beta)^2) = r_2(2\alpha\beta \times 10 + \beta^2)\,. \tag{6}$$

If x has more than two digits, then every digit to the left of the second digit has a factor of 10^2 or a great power of 10. Thus, everyone of those terms generated by one of those terms in the product x^2 will be eliminated by the function $r_2(x^2)$.

So, the x's we care about must have at least two digits, although we only care about the two rightmost digits. This takes us back to Eq. (5).

$$(\alpha\beta)^2 \to 2\alpha\beta, \beta^2\,. \tag{7}$$

Now, whatever $2\alpha\beta$ is, it's an even number. So, the only way to make it odd is to have an odd carry come from β^2. But for β^2 to have a carry at all, then β has to be in the set 4..9 and $p_1(\beta^2)$ is odd. But we already showed in (1) that no digit squared has both an odd remainder (mod 10) and an odd carry. Hence, we just proved that no positive integer x squared has both $p_1(x^2)$ and $p_2(x^2)$ in the odd digits.