

Math Diversion 1013

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A mathematician is a machine for turning coffee into theorems.

— Alfred Rényi

Source: <https://www.youtube.com/watch?v=pwkNnHVHVUc>

Title: The SECRET Property of ALL 2×2 matrices!

Presenter: Math Mastery with Amitesh

1 Introduction

I started off attempting this problem by using 2×2 matrices over a Pauli matrix basis. It quickly got too messy to show here (at least that's what I thought), so I asked Copilot what I should do. What follows is mostly the proof that Copilot produced. It's very elegant because it's based on the elegant Cayley-Hamilton theorem. Just the same, perhaps I'll return to this problem to solve it along the lines I had tried before.

2 Problem

Given two 2×2 matrices A, B with entries from a commutative ring with unity R , show that

$$(AB - BA)^2 = xI, \tag{1}$$

where $x \in R$ and I is the identity matrix for 2×2 matrices.

3 Preparation

Let $C := AB - BA$. For 2×2 matrices over a commutative ring R , the Cayley-Hamilton identity has the usual form

$$C^2 - (\operatorname{tr} C)C + (\det C)I = 0.$$

So if $\operatorname{tr} C = 0$, then automatically

$$C^2 = -(\det C)I,$$

which is exactly of the required form with $x = -(\det(AB - BA)) \in R$.

4 Solution

Trace of a commutator

Write $A = (a_{ij})$, $B = (b_{ij})$. Then

$$\operatorname{tr}(AB) = \sum_{i=1}^2 (AB)_{ii} = \sum_{i=1}^2 \sum_{k=1}^2 a_{ik} b_{ki} = \sum_{i,k} a_{ik} b_{ki}.$$

Similarly,

$$\operatorname{tr}(BA) = \sum_{i,k} b_{ik} a_{ki}.$$

Relabel indices $(i, k) \mapsto (k, i)$ in the second sum and use commutativity in R ($a_{ik} b_{ki} = b_{ki} a_{ik}$) to get $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Hence

$$\operatorname{tr}(C) = \operatorname{tr}(AB - BA) = 0.$$

5 Cayley–Hamilton for 2×2

For completeness: if $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then $\operatorname{tr} C = p + s$, $\det C = ps - qr$, and a direct multiplication shows

$$C^2 - (p + s)C + (ps - qr)I = 0,$$

which uses only commutativity of R .

6 Conclusion

Since $\operatorname{tr}(AB - BA) = 0$, Cayley–Hamilton gives

$$(AB - BA)^2 = -(\det(AB - BA))I.$$

So $(AB - BA)^2 = xI$ with $x = -(\det(AB - BA)) \in R$.