Math Diversion 1016

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People often overlook the obvious.
— Doctor Who

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Source: https://www.youtube.com/watch?v=6JN-DJuSFUY
Title: Renormalization and envelopes
 Presenter: Steven Strogatz
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1 Problem

In the beginning of his lecture on "Renormalization and envelopes" (Lecture 27), Steven Strogatz presents a figure similar to the following one, which is to indicate a continuously sliding line segment of fixed length (one unit for this case).

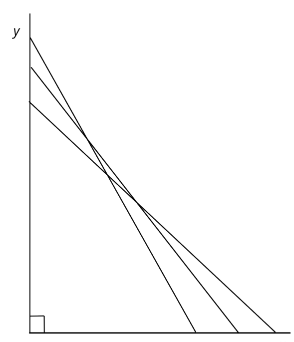

Figure 1. The continuously sliding line segment of fixed length (one unit).

We will show that there exists a curve in the xy-plane that each of these line segments is tangent to. Such a curve is called an "envelope" to the set of tangent lines. In the next figure we pick a typical line-segment tangent, showing a point (x, y) on it, which it shares with the envelope curve. The point of intersection (x, y) is parameterized by the angle τ.

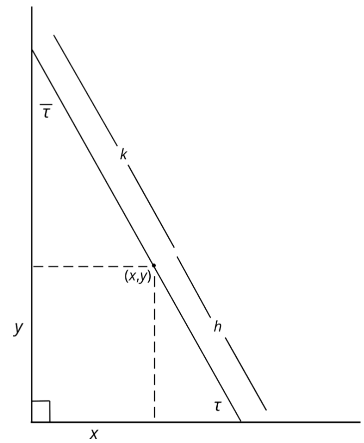

Figure 2. The continuously sliding line segment of fixed length (one unit). Note that $\bar{\tau} = 90^\circ - \tau$ is the complement to τ.

1) Show that the following relation holds between x and y:

$$\frac{x}{\cos\tau} + \frac{y}{\sin\tau} = k + h = 1\,. \tag{1}$$

After multiplying through by $\sin\tau\cos\tau$, we have that

$$x\sin\tau + y\cos\tau = \sin\tau\cos\tau\,. \tag{2}$$

Next, we apply a standard trick. When a set of points is defined implicitly, as in (2), we can define a constant function

$$F(x, y, \tau) = x\sin\tau + y\cos\tau - \sin\tau\cos\tau = 0\,. \tag{3}$$

Now, any differential change in coordinates x, y along the envelope defined by (3) will not change the value of F; hence

$$\frac{\partial F(x, y, \tau)}{\partial\tau} = 0\,, \tag{4}$$

which, when we work it out, becomes

$$x\cos\tau - y\sin\tau = \cos^2\tau - \sin^2\tau\,. \tag{5}$$

2) Use Eqs. (2) and (6) to show that the points (x, y) on the envelope satisfy the equation

$$x^{2/3} + y^{2/3} = 1\,. \tag{6}$$

2 Solution

So begin with the solution to Part 1.

For starters, we have that

$$k + h = 1\,. \tag{7}$$

Next, we have that

$$x = k\cos\tau\,, \quad y = h\cos\bar{\tau} = h\sin\tau\,. \tag{8}$$

On combining these, we get

$$\frac{x}{\cos\tau} + \frac{y}{\sin\tau} = 1\,. \tag{9}$$

Now we establish Part 2.

Now I'll collect our two simultaneous equations:

$$x\sin\tau + y\cos\tau = \sin\tau\cos\tau\,, \tag{10a}$$

$$x\cos\tau - y\sin\tau = \cos^2\tau - \sin^2\tau\,. \tag{10b}$$

What next? Multiply (10a) through by $\sin\tau$ and (10b) through by $\cos\tau$ and add these resulting equations together to get (with some trig simplification):

$$x = \cos^3\tau \quad \text{hence} \quad \cos\tau = x^{1/3}\,. \tag{11}$$

Then, multiply (10a) through by $\cos\tau$ and (10b) through by $\sin\tau$ and subtract these resulting equations (bottom from top) to get:

$$y = \sin^3\tau \quad \text{hence} \quad \sin\tau = y^{1/3}\,. \tag{12}$$

Finally, use these last results to eliminate the trig functions from (9), and we have that:

$$\frac{x}{x^{1/3}} + \frac{y}{y^{1/3}} = 1\,, \tag{13}$$

which simplifies to

$$x^{2/3} + y^{2/3} = 1\,. \tag{14}$$