

Math Diversion 1018

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Sooner or later, everyone comes
across a random matrix.

— P. Vivo

Source: <https://www.youtube.com/watch?v=6JN-DJuSFUY>

Title: Renormalization and envelopes

Presenter: Steven Strogatz

1 Problem

In the beginning of his lecture on “Renormalization and envelopes” (Lecture 27), Steven Strogatz presented a problem in the construction of an “envelope” to the set of tangent lines. We solved that problem in Problem 1016.

This time, we move to the next problem, which is to find the solution to the following boundary-value differential equation,

$$\epsilon y'' + (1 + \epsilon)y' + y = 0, \quad (1)$$

where the boundary conditions are

$$y(0) = 0 \quad \text{and} \quad y(1) = 1. \quad (2)$$

Show that the solution is

$$y(x, \epsilon) = \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}}. \quad (3)$$

2 Solution

We'll try the standard ansatz solution:

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}. \quad (4)$$

On substituting these into (1), we get

$$\epsilon r^2 e^{rx} + (1 + \epsilon)re^{rx} + e^{rx} = 0, \quad (5)$$

where r is to be determined. However, we know that $e^{rx} \neq$, therefore we can divide it out, yielding

$$\epsilon r^2 + (1 + \epsilon)r + 1 = 0. \quad (6)$$

Since this last equation is just a quadratic in r , we'll use the quadratic formula to solve for its roots (skipping a lot of boring details):

$$r = \frac{-(1 + \epsilon) \pm |1 - \epsilon|}{2\epsilon}. \quad (7)$$

So what to do about those absolute value signs? Demote them to parentheses and place a plus/minus sign in front. But we already have a plus/minus sign there, so we don't need to add them again. Hence,

$$r_+ = -1 \quad \text{and} \quad r_- = -\epsilon^{-1}. \quad (8)$$

So, we can write¹

$$\begin{aligned} y &= Ae^{r_+x} + Be^{r_-x} \\ &= Ae^{-x} + Be^{-x/\epsilon}, \end{aligned} \quad (9)$$

where A, B and constants which we need to determine. But how? Well, we have yet to use the boundary condition, and since there are two of them, that should be enough to do it.

On using that $y(0) = 0$, we have that

$$y(0) = 0 = A + B. \quad (10)$$

Therefore

$$B = -A. \quad (11)$$

On using that $y(1) = 1$, we have that

$$\begin{aligned} y(1) = 1 &= Ae^{-1} + Be^{-1/\epsilon} \\ &= Ae^{-1} + (-A)e^{-1/\epsilon} \\ &= A(e^{-1} - e^{-1/\epsilon}). \end{aligned} \quad (12)$$

Therefore

$$A = \frac{1}{e^{-1} - e^{-1/\epsilon}}, \quad (13)$$

and B is the negative of this. Using this information in (9), we have that

$$y(x, \epsilon) = \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}}. \quad (14)$$

¹I won't go into detail here, but since (1) is a linear differential equation, the arbitrary linear combinations of e^{r_+x} and e^{r_-x} must also be a solution.