Math Diversion 1025

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I love it when a plan comes together.
— Hannibal Smith, *The A-Team*

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Source: To be determined (hopefully)
Title: ?
 Presenter: Patrick
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1 Problem

Starting with x amount of pure liquid A in a beaker, we will n times repeat the following process: Draw out of the beaker y amount of the liquid (by volume) and add y amount of a different liquid Z to the beaker and let the contents come to a homogeneous mixture before repeating the cycle. Show that, at the nth cycle, the amount of the original liquid A is given by the formula

$$Q_n = x(1 - y/x)^n \,. \tag{1}$$

2 Solution

I'll use a standard induction proof for this formula, but before I get to that, I want to first motivate the formula.

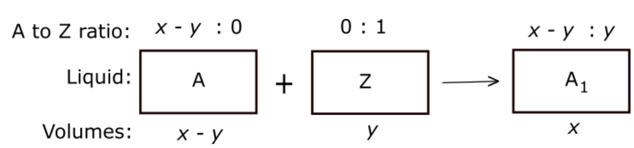

Figure 1. We'll consider the amount of liquid A in the beaker after the first cycle. The beaker state on the left shows the volume after the y quantity has been removed, but on the right, after the y amount of liquid Z has been added.

The amount of liquid A in mixture A_1 is given by

$$Q_1 = \frac{x-y}{x}(x) = x - y = x(1 - y/x)\,. \tag{2}$$

Let's do one more cycle. We now pour off y volume of liquid in the beaker.

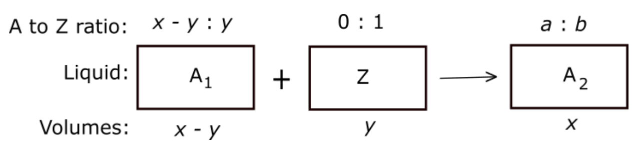

Figure 2. We're now on the second cycle.

The quantity of A in mixture A_2 is given by

$$Q_2 = \frac{a}{a+b}\,(x)\,. \tag{3}$$

To get $a/(a+b)$ we can balance this cycle for liquid A:

$$Q_2 = \frac{x-y}{x}(x-y) = \frac{a}{a+b}\,(x)\,. \tag{4}$$

That was easy. We can ignore a and b. Also, we just need to conform Q_2 to the pattern we're looking to prove.

$$Q_2 = \frac{x-y}{x}(x-y) = x(1-y/x)^2\,. \tag{5}$$

Now begins the induction proof of the formula (1). Clearly, the formula holds for $n = 0$, the base step. Next, we take as the inductive hypothesis, the assertion that

$$Q_k = x(1-y/x)^k\,, \tag{6}$$

and show that it holds for step $k+1$. That is, the last equation is still true if we replace k by $k+1$.

Now, at the end of the kth cycle, we have x amount of liquid, divided between Q_k amount of the original liquid A (by the inductive hypothesis) and the rest is liquid Z, of amount $x - Q_k$. This is presumed to be a homogeneous mixture, and then we draw off y amount from the beaker, and we're ready for the next step: the cycle that takes us from k to $k+1$:

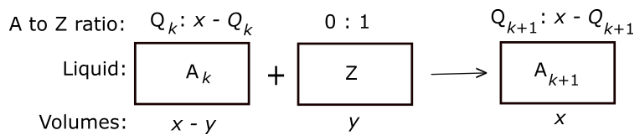

Figure 3. We're now on the $(k+1)$st cycle.

To recap: the end of the kth cycle left us with x amount of liquid in the beaker with Q_k amount of liquid A in liquid A_k. The starting point of the beaker in Figure 3 is right after we removed y amount of liquid A_k, leaving $x-y$ amount of liquid in the beaker.

So, balancing on liquid A across the cycle, we get

$$\frac{Q_k}{x}(x-y) + 0 = \frac{Q_{k+1}}{x}x\,. \tag{7a}$$

Simplifying, we get

$$Q_{k+1} = Q_k(1-y/x)\,. \tag{7b}$$

Using the inductive hypothesis (6) to substitute in for Q_k, we get

$$Q_{k+1} = x(1-y/x)^k \cdot (x-y) = x(1-y/x)^{k+1}\,. \tag{8}$$

And that's what we had to show.