Math Diversion 1051

P. Reany

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Not only is the Universe stranger than we think,
it is stranger than we can think.
— Werner Heisenberg

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Source:  The Ether of Great Mathematical Ideas
Title: To Lambert or not to Lambert?
 Presenter: Patrick
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1 Introduction

I've solved a lot of problems over the last two years that involve the Lambert W function, and I want to show you a couple insights I have from doing so. First, we know that $W(xe^x) = x$ is standard. But many of the problems I had to solve were of the form $xa^x = b$ where a is a positive real number and b is real. So, I proved a simple lemma that $W(xa^x) = W(b)$ can be written as

$$x = W_{(a)}(xa^x) = W_{(a)}(b) = \frac{W_n(b \ln a)}{\ln a}. \tag{1}$$

The use of the parentheses in the subscripts indicates a special meaning (i.e., that the W function is being evaluated base a, rather than the default base e) and not the using of the usual integers for n. Of course, when $a = e$, we have that.

$$x = W_{(e)}(xe^x) = W_{(e)}(b) = \frac{W_n(b \ln e)}{\ln e} = W_n(b). \tag{2}$$

Note: There is a write-up on the Lambert W function the reader may download that goes into depth on these lemmas used here.

2 Problem

Now, say the problem is to find a real solution to

$$x36^x = 3. \tag{3}$$

3 Solution

We can solve this for x by inspection and get $x = 1/2$. Anyway, to use the Lambert W function method, we do NOT first convert this problem in base a to base e. The lemma takes care of that for us, saving both time and space on the page! Then,

$$x = W_{(36)}(3) = \frac{W_n(3\ln 36)}{\ln 36}\,. \tag{4}$$

But wait! We already know that there is a real solution that does not involve the Lambert W function. We begin by taking the principal solution ($n = 0$):

$$x_0 = \frac{W(3\ln 36)}{\ln 36}, \tag{5}$$

and now we extricate $3\ln 36$ from the argument to $W(\cdot)$. And this is doable.

$$\frac{W(3\ln 36)}{\ln 36} = \frac{W(3\ln 6^2)}{\ln 6^2} = \frac{W(6\ln 6)}{2\ln 6} = \frac{\ln 6}{2\ln 6} = \frac{1}{2}, \tag{6}$$

where I used the lemma

$$W(z\ln z) = \ln z\,. \tag{7}$$

Hence, a real solution to (3) is

$$x_0 = \frac{1}{2}\,. \tag{8}$$

The point of this demonstration being that it's possible to use the Lambert W function to solve certain problems, even when you know the final answer does not contain the Lambert W function — you just have to find the way to extricate out of it.