Math Diversion 1058

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In the middle of difficulty lies opportunity.
— John Archibald Wheeler

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Source:  https://www.youtube.com/watch?v=rfArXK422FM
Title: Russian Math Olympiad Questions
 Presenter: Math Beast
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1 Problem

Given the relation

$$a^a = 2^{\sqrt{200}}\,, \tag{1}$$

solve for real values of a.

I choose to rewrite the Given as

$$a^a = 2^{10\sqrt{2}}\,, \tag{2}$$

2 Solution #1

In this solution, we try the α-transformation. Let

$$a = 2^\alpha\,. \tag{3}$$

After substituting this into (2), we get

$$2^{\alpha 2^\alpha} = 2^{10\sqrt{2}}\,, \tag{4}$$

On equating exponents, we have that

$$\alpha 2^\alpha = 10\sqrt{2} = 5 \times 2^{3/2}\,. \tag{5}$$

Next, I'll set

$$\alpha = 5\beta\,, \tag{6}$$

and substitute this into the previous equation:

$$(5\beta)2^{5\beta} = 5 \times 2^{3/2}\,, \tag{7}$$

and this simplifies to

$$\beta 2^{5\beta} = 2^{3/2}\,. \tag{8}$$

We can solve this by inspection to get

$$\beta = 1/2\,. \tag{9}$$

On substituting this into (6), we have that

$$\alpha = 5/2\,, \tag{10}$$

Thus, we get for a:

$$a = 2^{5/2} = 4\sqrt{2}\,. \tag{11}$$

3 Solution #2

For this solution, I'll use the Lambert W function and the identity:

$$W(x\ln x) = \ln x\,. \tag{12}$$

So, on taking the natural logarithm of (2), we get

$$a\ln a = 10\sqrt{2}\ln 2\,. \tag{13}$$

At this point I would routinely take the Lambert W function across this equation, to get

$$\ln a = W(10\sqrt{2}\ln 2) = W(5\times 2\times\sqrt{2}\ln 2)\,. \tag{14}$$

We know from the first solution, that we don't need to use the Lambert W function to present the solution to this problem. Therefore, there must be some way to 'extricate' out of the Lambert W function. To this purpose, perhaps we could use the identity (12) once more, though if I had not known that the solution did not need the Lambert W function, I might not bother to look for it.

Anyway, I think I should proceed by replacing the argument of the logarithm by $\sqrt{2}$:

$$\begin{aligned}\ln a &= W(5\times 4\times \tfrac{1}{2}\times\sqrt{2}\ln 2)\\ &= W(5\times 4\times\sqrt{2}\ln\sqrt{2})\\ &= W(4\times\sqrt{2}\ln(\sqrt{2})^5)\\ &= W(4\sqrt{2}\ln(4\sqrt{2})) \qquad (15)\\ &= \ln 4\sqrt{2}\,. \qquad (16)\end{aligned}$$

So, even though we entered the realm of the Lambert W function to solve for a, we used an identity that allowed us to extricate ourselves out of it. Then, solving for a, we have that

$$a = 4\sqrt{2}\,. \tag{17}$$