Math Diversion 1066

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Since I was a young child, I have
intuitively believed three things:
1) I live in a moral universe,
created by a moral Creator,
2) Racism is irrational,
3) Mean people suck.
— Patrick

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Source:  https://www.youtube.com/watch?v=Qc2u8licPD4
Title: Derivative of an Infinite Power of x
 Presenter: Brain Station Advanced
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1 Problem

Given the relation

$$\phi(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}, \tag{1}$$

solve for all real values of x.

2 Solution

The function ϕ is said to be self-similar because a "part" of the expression it represents is similar to the whole. Use this fact to understand the next step:

$$\phi(x) = x^{\phi(x)} . \tag{2}$$

The next step involves looking at a general case of one function to the power of another function, such as

$$h(x) = f(x)^{g(x)} . \tag{3}$$

Now, if we want to differentiate this equation to get $h'(x)$, the usual procedure is to take the natural logarithm of (3) before differentiating, to get

$$\ln h(x) = g(x) \ln f(x) . \tag{4}$$

So, why would we do this? Because if we know how to differentiate a natural logarithm and how to use the product rule, we're ready to differentiate. So, let's do it!

$$\frac{h'}{h} = g' \ln f + g \frac{f'}{f} \,. \tag{5}$$

We can use this last equation as a formula to substitute into. On comparing the functions of (4) to those of (2), we have that

$$h(x) = \phi(x) \,, \quad f(x) = x \,, \quad g(x) = \phi(x) \,. \tag{6}$$

This yields

$$\frac{\phi'}{\phi} = \phi' \ln x + \phi \frac{1}{x} \,. \tag{7}$$

Solving this for ϕ', we have that

$$\phi' = \frac{\phi^2}{x(1 - \phi \ln x)} \,. \tag{8}$$