# Math Diversion 1074

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The first thing you notice about Cedric Villani
is that he is wearing the wrong century.
— Turing, "Cedric Villani, The Most
Charismatic Man in Mathematics"

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Source: The Ether of Great Mathematical Ideas
Title: The Adjoint of the Pseudoscalar.
 Presenter: Patrick
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## 1 Preliminaries

Let $\mathcal{V}$ be a vector space of dimension $n$. Let $\mathcal{G}(\mathcal{V})$ be the geometric algebra over $\mathcal{V}$. Let $f$ be a linear function on $\mathcal{V}$.

The so-called *adjoint* of $f$, $\bar{f}$, is defined implicitly by the relation

$$x \cdot f(y) = \bar{f}(x) \cdot y \quad \text{for all } x, y \in \mathcal{V}\,. \tag{1}$$

Let $\underline{f}$ (the *differential* of $f$) be the extension of $f$ to apply to all *multivectors* of the geometric algebra $\mathcal{G}(\mathcal{V})$. We demand that $\underline{f}$ be a linear function over its space of multivectors. Let vector $a$ be in $\mathcal{V}$. Then $\underline{f}(a) = f(a)$. That is, the differential of $f$ treats the vectors of $\mathcal{V}$ the same as does $f$.

And to *bivectors*, such as $a \wedge b$, we get

$$\underline{f}(a \wedge b) = \underline{f}(a) \wedge \underline{f}(b)\,, \tag{2}$$

and similar expansions are to be made for trivectors and above. David Hestenes refers to this process of "distributing" $\underline{f}$ over a serial wedge product the "outermorphism." The trivector example is:

$$\underline{f}(a \wedge b \wedge c) = \underline{f}(a) \wedge \underline{f}(b) \wedge \underline{f}(c)\,, \tag{3}$$

and so on.

That leaves the question of how we should define the differential of a scalar, say $\lambda$. Scalars of a geometric algebra are real numbers, and they pass through, so to speak, unaffected, that is:

$$\underline{f}(\lambda) = \lambda. \tag{4}$$

So, what happens when we apply this differential to the pseudoscalar $I$ of $\mathcal{G}(\mathcal{V})$? We'll solve this by a definition:

$$\underline{f}(I) = (\det f)I\,, \tag{5}$$

where $\det f$ is the determinant of the linear transformation $f$, and it's a scalar.

## 2 Problem

Assuming that $\bar{f}$ is itself a linear, outermorphism, show that

$$\bar{f}(I) = (\det f)I\,, \tag{6}$$

which might seem counterintuitive.

## 3 Proof

We begin with the identity (from *Clifford Algebra to Geometric Calculus* by Hestenes and Sobczyk):

$$A_r \cdot \bar{f}(B_s) = \bar{f}[\underline{f}(A_r) \cdot B_s] \quad \text{where } r \leq s. \tag{7}$$

Now, set $r = s = n$, where $n$ is the dimension of the original vector space $\mathcal{V}$. And then set $A_r = B_s = I$, Then

$$I \cdot \bar{f}(I) = \bar{f}[\underline{f}(I) \cdot I], \tag{8}$$

or

$$I\bar{f}(I) = \bar{f}[(\det f)II] = (\det f)II\,, \tag{9}$$

since '$(\det f)II$' is a scalar.[1] Anyway, dividing out one factor of $I$ on both sides, we get

$$\bar{f}(I) = (\det f)I. \tag{10}$$

Done.

[1]The square of a (unit) pseudoscalar $I$ is $\pm 1$, that is $I^2 = \pm 1$.