

Math Diversion 1075

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Rest is medicine.
— Medical Adage

Source: The Ether of Great Mathematical Ideas

Title: Inverses of Linear Transformations.

Presenter: Patrick

1 Introduction

Let f be a nonsingular linear transformation from an n -dimensional vector space \mathcal{V} to itself. To keep things simple, by nonsingular I just mean that the determinant of the transformation is nonzero, which I'll write as $\det f \neq 0$.

Since this little article is not a development of the details of geometric algebra, I must refer the reader to my many papers on geometric algebra or else (even better) consult the two sources *New Foundations for Classical Mechanics* [1] and *Clifford Algebra to Geometric Calculus* [2].

2 Preliminaries

Let \mathcal{V} be a vector space of dimension n . Let $\mathcal{G}(\mathcal{V})$ be the geometric algebra over \mathcal{V} . Let f be a linear function on \mathcal{V} .

The so-called *adjoint* of f , \bar{f} , is defined implicitly by the relation

$$x \cdot f(y) = \bar{f}(x) \cdot y \quad \text{for all } x, y \in \mathcal{V}. \quad (1)$$

Let \underline{f} (the *differential* of f) be the extension of f to apply to all *multivectors* of the geometric algebra $\mathcal{G}(\mathcal{V})$. We demand that \underline{f} be a linear function over its space of multivectors. Let vector a be in \mathcal{V} . Then $\underline{f}(a) = f(a)$. That is, the differential of f treats the vectors of \mathcal{V} the same as does f .

And to *bivectors*, such as $a \wedge b$, we get

$$\underline{f}(a \wedge b) = \underline{f}(a) \wedge \underline{f}(b), \quad (2)$$

and similar expansions are to be made for trivectors and above. David Hestenes refers to this process of “distributing” \underline{f} over a serial wedge product the “outermorphism.” The trivector example is:

$$\underline{f}(a \wedge b \wedge c) = \underline{f}(a) \wedge \underline{f}(b) \wedge \underline{f}(c), \quad (3)$$

and so on.

That leaves the question of how we should define the differential of a scalar, say λ . Scalars of a geometric algebra are real numbers, and they pass through, so to speak, unaffected, that is:

$$\underline{f}(\lambda) = \lambda. \quad (4)$$

So, what happens when we apply this differential to the pseudoscalar I of $\mathcal{G}(\mathcal{V})$?¹ We’ll solve this by a definition:

$$\underline{f}(I) = (\det f)I, \quad (5)$$

where $\det f$ is the determinant of the linear transformation f , and it’s a scalar.

Assuming that \bar{f} is itself a linear, outermorphism,² show that

$$\bar{f}(I) = (\det f)I, \quad (6)$$

which might seem counterintuitive.

We begin with the identity (from *Clifford Algebra to Geometric Calculus* by Hestenes and Sobczyk):

$$A_r \cdot \bar{f}(B_s) = \bar{f}[f(A_r) \cdot B_s] \quad \text{where } r \leq s. \quad (7)$$

Now, set $r = s = n$, where n is the dimension of the original vector space \mathcal{V} . And then set $A_r = B_s = I$, Then

$$I \cdot \bar{f}(I) = \bar{f}[f(I) \cdot I], \quad (8)$$

or

$$I\bar{f}(I) = \bar{f}[(\det f)II] = (\det f)II, \quad (9)$$

since ‘ $(\det f)II$ ’ is a scalar.³ Anyway, dividing out one factor of I on both sides, we get

$$\bar{f}(I) = (\det f)I. \quad (10)$$

Done.

¹The pseudoscalar I of $\mathcal{G}(\mathcal{V})$ can be formed by taking the wedge product of all the vectors of an orthonormal basis for \mathcal{V} .

²For scalar λ , $\bar{f}(\lambda) = \lambda$.

³The square of a (unit) pseudoscalar I is ± 1 , that is $I^2 = \pm 1$.

3 Problem:

My goal is to prove that $ff^{-1} = 1$ and that $f^{-1}f = 1$ by showing that for arbitrary x of vector space \mathcal{V} , that

$$ff^{-1}x = x \quad \text{and} \quad f^{-1}fx = x. \quad (11)$$

So, let I be the unit pseudoscalar of the geometric algebra \mathcal{G}_n associated with \mathcal{V} . Then whenever we see $A \cdot I$ for any A we like, we can replace it by AI .

I'll need two results from *Clifford Algebra to Geometric Calculus*, pg. 69, Eqs. (1.14a,b):

$$A_r \cdot \bar{f}(B_s) = \bar{f}[f(A_r) \cdot B_s] \quad \text{for } r \leq s, \quad (12)$$

$$\underline{f}(A_r) \cdot B_s = \underline{f}[A_r \cdot \bar{f}(B_s)] \quad \text{for } r \geq s. \quad (13)$$

This last equation can be put into the more convenient form for use here as:

$$\underline{f}[\bar{f}(B_s) \cdot A_r] = B_s \cdot \underline{f}(A_r) \quad \text{for } r \geq s. \quad (14)$$

There are two more results that'll be handy to know. First, for general multivector A ,

$$\underline{f}^{-1}A = \frac{\bar{f}(AI)I^{-1}}{\det f}. \quad (15)$$

And second

$$\underline{f}(I) = \bar{f}(I) = (\det f)I. \quad (16)$$

And since I^{-1} and I can only differ by a scale factor and since \underline{f} is linear, then

$$\underline{f}(I^{-1}) = \bar{f}(I^{-1}) = (\det f)I^{-1}. \quad (17)$$

4 Proof:

If we restrict the arguments of f and f^{-1} to vectors, then

$$\underline{f}x = fx = f(x), \quad (18)$$

$$\underline{f}^{-1}x = f^{-1}x = f^{-1}(x). \quad (19)$$

Now, let's replace the blade A in (15) by a simple vector $y \in \mathcal{V}$, to get

$$f^{-1}y = \underline{f}^{-1}y = \frac{\bar{f}(yI)I^{-1}}{\det f}. \quad (20)$$

Then, we hit both sides of this by f :

$$\begin{aligned}
f(f^{-1}y) &= f\left(\frac{\bar{f}(yI)I^{-1}}{\det f}\right) \\
&= \frac{1}{\det f} f(\bar{f}(yI)I^{-1}) \quad (\text{linearity of } f) \\
&= \frac{1}{\det f} \underline{f}(\bar{f}(yI)I^{-1}) \quad (\text{by (18) as } \bar{f}(yI)I^{-1} \text{ is a vector}) \\
&= \frac{1}{\det f} (yI) \cdot \underline{f}(I^{-1}) \quad (\text{by (14)}) \\
&= (yI) \cdot (I^{-1}) = \langle (yI)(I^{-1}) \rangle_1 \\
&= yII^{-1} \\
&= y. \tag{21}
\end{aligned}$$

Now, for the second part of the proof. This time, we go back to (20) and replace y by $\underline{f}(x)$:

$$\begin{aligned}
f^{-1}f(x) &= \underline{f}^{-1}\underline{f}(x) = \frac{\bar{f}(\underline{f}(x) \cdot I)I^{-1}}{\det f} \quad (\text{by (15)}) \\
&= x \cdot \bar{f}(I)I^{-1} \frac{1}{\det f} \quad (\text{by (12)}) \\
&= x(\det f)II^{-1} \frac{1}{\det f} \\
&= x. \tag{22}
\end{aligned}$$

References

- [1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers (1999).
- [2] D. Hestenes, and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Reidel (1987).