Math Diversion 1079

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Learning is a treasure that will follow its owner everywhere.
— Chinese proverb

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Source:  https://www.youtube.com/watch?v=0sGS1yVAoHE
Title: A Clean Trick to Solve This
 Presenter: Sybermath
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1 Problem

Given the relation

$$x = 2^{x-2}\,, \tag{1}$$

find the two real values of x.

Note: See my write-up on the Lambert W function for the Lambert W function base B.

2 Solution

Right off the bat, by inspection, I see that $x = 4$ is a real solution.

Anyway, this looks like a job for the Lambert W function. My first step is to transform (1) into

$$x = 2^x \frac{1}{4}\,. \tag{2}$$

Next, to

$$x2^{-x} = \frac{1}{4}\,. \tag{3}$$

And now to

$$(-x)2^{-x} = -\frac{1}{4}\,. \tag{4}$$

Now I take the Lambert W function base 2 across this equation, to get

$$-x = W_{(2)}(-\tfrac{1}{4}) = \frac{W_n(-\frac{1}{4}\ln 2)}{\ln 2} \quad \text{for } n \in \mathbb{Z}\,. \tag{5}$$

Thus, I get for the general solution

$$x = -\frac{W_n(-\frac{1}{4}\ln 2)}{\ln 2} \quad n \in \mathbb{Z}\,, \tag{6}$$

as also does WolframAlpha. Using some standard algebra, this becomes

$$x = \frac{W_n(\frac{1}{4}\ln\frac{1}{2})}{\ln\frac{1}{2}} \quad n \in \mathbb{Z}\,. \tag{7}$$

So, to use this formula to get the two real solutions (which WolframAlpha claims to be $x = 4$ and $x \approx 0.309907$), I have to assume that they follow from

$$x_{-1,0} = \frac{W_{-1,0}(\frac{1}{4}\ln\frac{1}{2})}{\ln\frac{1}{2}} \quad n \in \mathbb{Z}\,. \tag{8}$$

So, to get the value $x = 4$, I need to extricate out of the Lambert W function. I assume that means the principal value, given by

$$x_0 = \frac{W_0(\frac{1}{4}\ln\frac{1}{2})}{\ln\frac{1}{2}} \quad n \in \mathbb{Z}\,. \tag{9}$$

But I only have the following three common "extrication" formulas that seem appropriate to try:

$$W_0(x^{x+1}\ln x) = x\ln x\,, \tag{10}$$
$$W_0(x\ln x) = \ln x\,, \tag{11}$$
$$W_0(xe^x) = x\,. \tag{12}$$

But I couldn't figure out how to make any of them work.

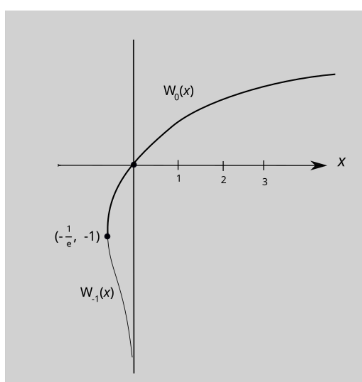

Figure 1. A sketch of the Lambert W function near the origin.

From the figure, we can see that the Lambert W function can yield real values on negative values between 0 and $-\frac{1}{e}$. W_0 is called the *principal* value of Lambert.

Let's go back to (6). To get the value $x = 4$, we need to have[1]

$$\frac{W_{-1}(-\frac{1}{4}\ln 2)}{\ln 2} = -4\,. \tag{13}$$

So, this must be correct, but how to prove it's correct? One way would be to use the approximation formulas to calculate it directly. That's not a very appealing method, when one knows the answer is an integer. The usual method I have employed successfully in the past has been to use one of the extrication formulas above. But as I said, I couldn't figure how to accomplish that this time.

So, I turned to outside help on this one. I explained the situation to this point to Copilot. Apparently Copilot could not succeed by using one of the extrication formulas either, so it tried the following method instead.

Consider the identity

$$W(x)e^{W(x)} = x\,. \tag{14}$$

First, it converted the form to

$$We^{W} = -\tfrac{1}{4}\ln 2\,, \tag{15}$$

and tried the ansatz

$$W = -a\ln 2\,, \tag{16}$$

where a is to be determined. So let's substitute in:

$$(-a\ln 2)e^{(-a\ln 2)} = -\tfrac{1}{4}\ln 2\,, \tag{17}$$

which simplifies to

$$(-a\ln 2)2^{-a} = -\tfrac{1}{4}\ln 2\,, \tag{18}$$

which, after a bit of algebra, becomes

$$a = 2^{a-2}\,. \tag{19}$$

But this is just (1) all over again. ;)

So, in the end, neither Copilot nor I found a slick way to find the solution $x = 4$. By the way, Copilot agreed with WolframAlpha on the other real solution $x_0 \approx 0.309907$.

[1] We won't get -4 using the principal value W_0 because it ranges only from -1 up to 0.