Math Diversion 1082

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May 2, 2026

The definition of a good mathematical problem
is the mathematics it generates
— Andrew Wile

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Source:  https://www.youtube.com/watch?v=xBpZRWCGw30
Title: Is this equation solvable?
 Presenter: blackpenredpen
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1 Problem

Given the relation

$$x^{\ln 4} + x^{\ln 10} = x^{\ln 25}\,, \tag{1}$$

find the positive real solutions for x.

2 Solution

We start by making the substitution,

$$x = e^{\alpha}\,. \tag{2}$$

getting

$$e^{\alpha \ln 4} + e^{\alpha \ln 10} = e^{\alpha \ln 25}\,, \tag{3}$$

or

$$e^{\ln 4^{\alpha}} + e^{\ln 10^{\alpha}} = e^{\ln 25^{\alpha}}\,, \tag{4}$$

which, after simplifying becomes,

$$4^{\alpha} + 10^{\alpha} = 25^{\alpha}\,. \tag{5}$$

On dividing through by 25^{α}, we get

$$\left(\frac{4}{25}\right)^{\alpha} + \left(\frac{10}{25}\right)^{\alpha} = 1\,, \tag{6}$$

which simplifies to

$$\left(\frac{2}{5}\right)^{2\alpha} + \left(\frac{2}{5}\right)^{\alpha} - 1 = 0\,. \tag{7}$$

After a little algebra, we have that

$$\left[\left(\frac{2}{5}\right)^{\alpha}\right]^{2} + \left(\frac{2}{5}\right)^{\alpha} - 1 = 0\,. \tag{8}$$

Next, we make a variable substitution,

$$y = \left(\frac{2}{5}\right)^{\alpha}\,, \tag{9}$$

which brings (8) to the simple quadratic form

$$y^2 + y - 1 = 0\,, \tag{10}$$

which has solutions

$$y_{\pm} = \frac{-1 \pm \sqrt{5}}{2}\,. \tag{11}$$

We can solve for $\alpha_{\pm}$ in (9), to get

$$\alpha_{\pm} = \frac{\ln y_{\pm}}{\ln 2/5} = \frac{\ln\left(\frac{-1\pm\sqrt{5}}{2}\right)}{\ln 0.4} \approx \frac{\ln\left(\frac{-1\pm\sqrt{5}}{2}\right)}{(-0.9163)}\,. \tag{12}$$

Now,

$$y_{+} \approx 0.6180, \quad y_{-} \approx -1.6180\,, \tag{13}$$

which gives us that

$$\alpha_{+} \approx \frac{\ln 0.6180}{(-0.9163)} \approx 0.5253\,, \tag{14}$$

$$\alpha_{-} \approx \frac{\ln(-1.6180)}{(-0.9163)}\,. \tag{15}$$

We reject α_{-} because we will not allow the logarithm of a negative number in this solution.

So, to get x_{+}, we return to (2), to get

$$x_{+} \approx e^{0.5253} \approx 1.69097\,. \tag{16}$$

WolframAlpha gets

$$x_{+} \approx 1.69075\,. \tag{17}$$