# Math Diversion 1096

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He who would ride two camels, finds he can ride neither.
— From an old movie

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Source: https://www.youtube.com/watch?v=6ALDtCgeXhg
Title: This AP Calculus BC Integral Looks Impossible
 Presenter: Dr PK Math
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## 1 Problem

Given the relation

$$I = \int_0^1 \frac{5x+8}{3x^2+2x+2}\, dx\,, \tag{1}$$

solve for $I$.

## 2 Solution

The basic idea for this solution, at least the way I intend to do it, is to transform the integrand to conform to two familiar integrands, the first being

$$\int \frac{dY}{Y} = \ln|\,Y\,| + C\,, \tag{2}$$

and the second being

$$\int \frac{dZ}{Z^2+1} = \tan^{-1} Z + C\,. \tag{3}$$

So, let's begin.

If

$$Y = 3x^2 + 2x + 2\,, \tag{4}$$

from (1), then

$$dY = (6x+2)dx\,. \tag{5}$$

So, the trick is to replace the $5x$ in the numerator of (1) with a $6x$. This requires just a bit of arithmetic, thus

$$\begin{aligned} I &= \frac{5}{6}\frac{6}{5}\int_0^1 \frac{5x+8}{3x^2+2x+2}\,dx = \frac{5}{6}\int_0^1 \frac{6x+48/5}{3x^2+2x+2}\,dx \\ &= \frac{5}{6}\int_0^1 \frac{6x+2+38/5}{3x^2+2x+2}\,dx \\ &= \frac{5}{6}\int_0^1 \frac{6x+2}{3x^2+2x+2}\,dx + \frac{5}{6}\int_0^1 \frac{38/5}{3x^2+2x+2}\,dx\,. \end{aligned} \tag{6}$$

We already know how to integrate the first term:

$$\frac{5}{6}\int_0^1 \frac{6x+2}{3x^2+2x+2}\,dx = \frac{5}{6}\ln\left|\,3x^2+2x+2\,\right|\Big|_0^1 = \frac{5}{6}\ln 7/2\,. \tag{7}$$

That leaves us with the second term to integrate. So, let

$$J = \int_0^1 \frac{38/5}{3x^2+2x+2}\,dx\,. \tag{8}$$

Then

$$\begin{aligned} J &= \frac{1/3}{1/3}\int_0^1 \frac{38/5}{3x^2+2x+2}\,dx = \int_0^1 \frac{38/15}{x^2+\frac{2}{3}x+\frac{2}{3}}\,dx \\ &= \frac{38}{15}\int_0^1 \frac{1}{(x+\frac{1}{3})^2-\frac{1}{9}+\frac{2}{3}}\,dx = \frac{38}{15}\int_0^1 \frac{1}{(x+\frac{1}{3})^2+\frac{5}{9}}\,dx \\ &= \frac{38}{15}\int_0^1 \frac{9/5}{\frac{9}{5}(x+\frac{1}{3})^2+1}\,dx = \frac{38}{15}\frac{9}{5}\int_0^1 \frac{1}{[\frac{3}{\sqrt{5}}(x+\frac{1}{3})]^2+1}\,dx \\ &= \frac{38}{15}\frac{9}{5}\int_0^1 \frac{\frac{\sqrt{5}}{3}}{[\frac{3}{\sqrt{5}}(x+\frac{1}{3})]^2+1}\,d[\tfrac{3}{\sqrt{5}}(x+\tfrac{1}{3})] \\ &= \frac{38}{15}\frac{9}{5}\frac{\sqrt{5}}{3}\int_0^1 \frac{1}{[\frac{3}{\sqrt{5}}(x+\frac{1}{3})]^2+1}\,d[\tfrac{3}{\sqrt{5}}(x+\tfrac{1}{3})] = \frac{38}{5\sqrt{5}}\int_0^1 \frac{1}{[\frac{3}{\sqrt{5}}(x+\frac{1}{3})]^2+1}\,d[\tfrac{3}{\sqrt{5}}(x+\tfrac{1}{3})] \\ &= \frac{38}{5\sqrt{5}}\tan^{-1}\tfrac{3}{\sqrt{5}}(x+\tfrac{1}{3})\Big|_0^1 \\ &= \frac{38}{5\sqrt{5}}[\tan^{-1}\tfrac{4}{\sqrt{5}}-\tan^{-1}\tfrac{1}{\sqrt{5}}]\,. \end{aligned} \tag{9}$$

Therefore

$$\begin{aligned} I &= \frac{5}{6}\ln 7/2 + \frac{5}{6}\frac{38}{5\sqrt{5}}\left[\,\tan^{-1}\tfrac{4}{\sqrt{5}}-\tan^{-1}\tfrac{1}{\sqrt{5}}\right] \\ &= \frac{5}{6}\ln 7/2 + \frac{19}{3\sqrt{5}}\left[\,\tan^{-1}\tfrac{4}{\sqrt{5}}-\tan^{-1}\tfrac{1}{\sqrt{5}}\right]. \end{aligned} \tag{10}$$