

Math Diversions, Problem 1

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1 Problem

This problem is found on the YouTube channel **PreMath**, from December 25, 2021. My solution here is a little different from that given by the presenter.

Statement of the problem:

Given the relation

$$x + \frac{1}{x} = 5, \quad (1)$$

find the value of

$$x^5 + \frac{1}{x^5} = ? \quad (2)$$

Comment: Eq. (1) is a quadratic equation in x , and as such we can solve for x by use of the quadratic formula. But here we will solve for the value of the expression in (2) without first solving for x .

Solution to the problem:

Note that

$$\left(x + \frac{1}{x}\right)^2 = 25, \quad (3)$$

which reduces to

$$x^2 + \frac{1}{x^2} = 23. \quad (4)$$

Using Eq. (1), we have that

$$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) = 5 \cdot 23 = 115. \quad (5)$$

The LHS of this last equation expands to

$$x^3 + \frac{1}{x^3} + \left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3} + 5. \quad (6)$$

Thus (5) reduces to

$$x^3 + \frac{1}{x^3} = 110. \quad (7)$$

On multiplying (4) by (7), we get that

$$\left(x^2 + \frac{1}{x^2}\right)\left(x^3 + \frac{1}{x^3}\right) = 23 \cdot 110 = 2530. \quad (8)$$

And this partially reduces to

$$\left(x^5 + \frac{1}{x^5}\right) + \left(x + \frac{1}{x}\right) = 2530. \quad (9)$$

One more simplification gives us the result we're looking for:

$$x^5 + \frac{1}{x^5} = 2525. \quad (10)$$

Done.

Comment: It might be interesting to work the problem backwards, by which I mean to start with (10) and then find the value of $x + 1/x$.