Math Diversions, Problem 10

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1 Problem

This problem can be found at the YouTube video

`https://www.youtube.com/watch?v=0lIAF3DRVTw`

Statement of the problem:

Solve for all solutions a, b, c over the natural numbers to the relation

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=2\,. \tag{1}$$

This problem was given to the 1995 BMO[1] contestants. Presenter *Prime Newtons* provides an interesting solution to this problem. However, I have to admit that since this area of math isn't all that familiar to me, I hesitated to attempt it on my own seriously. So, instead, I tried to solve a related, simpler problem (think of this as a warmup problem), which is this:

Solve for one solution a, b, c over the integers the relation (1). The advantage of having all the integers to choose from is that one is allowed to use negative numbers.

2 A Solution

My intuition lead me to try to solve something of the form $AB = 1$, since I have intuitions about finding values that have to be multiplicative inverses to each other. Following that line of thought, let's set $c = 1$.[2] Then (1) reduces to

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)=1\,. \tag{2}$$

[1]BMO stands British Mathematical Olympiad.

[2]I refer to this kind of rash maneuver as a 'mad-dash ansatz'.

It's kind of obvious that we should multiply out the LHS and then simplify.[3]

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 0\,. \tag{3}$$

There's now an obvious trick to try here that I've used many times before in similar situations, which is to introduce the ratio of a and b and work with that. Therefore, let $\lambda \equiv a/b$. Now, multiply (3) through by b, and then smplify:

$$\frac{1}{\lambda} + 1 + \frac{1}{a} = 0\,. \tag{4}$$

Although it's true that we still have two variables to deal with, this equation looks easier to work with than the one prior. Clearly, we have to choose one of a or λ as a negative integer. The obvious choice to me is $a = -2$. This gives us

$$\frac{1}{\lambda} + \frac{1}{2} = 0\,. \tag{5}$$

And this forces $\lambda = -2$, and then $b = 1$.

Anyway, listing the solution, we have

$$a = -2, \quad b = 1, \quad c = 1\,. \tag{6}$$

Have fun trying the original problem!

[3] Alternatively, you might have found the solution at this point by inspection. But if you didn't, we go on.