Math Diversion Problem 101

P. Reany

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> In mathematics, the art of proposing a question must be held of higher value than solving it.
> — Georg Cantor

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=S64w7YxWoTs
Title: France | Olympic math question - past exam
      SAT MATH QUESTION
 Presenter: Kmath addict
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1 The Problem

Given the relation

$$5^x + 35^{x/2} = 7^x \,, \tag{1}$$

find the values of x.

2 The Solution

We begin by converting the given equation to

$$35^{x/2} = 7^x - 5^x \,, \tag{2}$$

and then squaring

$$35^x = 7^x \cdot 5^x = (7^x - 5^x)^2 \tag{3}$$

$$= (7^x + 5^x)^2 - 4 \cdot 7^x \cdot 5^x \,. \tag{4}$$

Now, we adjust:

$$5 \cdot 7^x \cdot 5^x = (7^x + 5^x)^2 \,. \tag{5}$$

It may not seem intuitive to take the square root on both sides, but that's the way to a quick finish.

$$\sqrt{5} \cdot 7^{x/2} \cdot 5^{x/2} = 7^x + 5^x \,. \tag{6}$$

But we can adjust the LHS by going back to (2):

$$\sqrt{5}(7^x - 5^x) = 7^x + 5^x \,. \tag{7}$$

Now we collect on 7^x and 5^x.

$$(\sqrt{5} - 1)7^x = (\sqrt{5} + 1)5^x \,, \tag{8}$$

which gives us

$$\frac{7^x}{5^x} = \left(\frac{7}{5}\right)^x = \frac{\sqrt{5}+1}{\sqrt{5}-1} \,. \tag{9}$$

On taking the logarithm of both sides, we get

$$x = \frac{\log \dfrac{\sqrt{5}+1}{\sqrt{5}-1}}{\log\left(\dfrac{7}{5}\right)} = \frac{\log \dfrac{(\sqrt{5}+1)^2}{4}}{\log\left(\dfrac{7}{5}\right)} = \frac{2\log \frac{1}{2}(\sqrt{5}+1)}{\log\left(\dfrac{7}{5}\right)} \,. \tag{10}$$