Math Diversion Problem 104

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> I love it when a plan comes together.
> — Hannibal Smith, *The A-Team*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=ZlDFHJqUziE
Title: Solving a nice integral - Solution to past exam
 Presenter: Kmath addict
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1 The Problem

Find the value of the following definite integral:

$$\int_0^1 \ln(1+x^2)\,dx\,. \tag{1}$$

2 The Solution

My plan is to use the standard identity

$$\int \ln w\,dw = w(\ln w - 1) + c\,, \tag{2}$$

in this solution.

Notation:

$$\bar{I} \equiv \int_0^1 \ln(1+x^2)\,dx\,, \tag{3}$$

but

$$I \equiv \int \ln(1+x^2)\,dx\,, \tag{4}$$

with the intent that I will work mostly with the indefinite integral I (ignoring the arbitrary constant) until the final steps.

$$I = \int \ln(1+x^2)\,dx = \int \ln\,(1+ix)(1-ix)\,dx \tag{5a}$$

$$= \int \ln\,(1+ix)\,dx + \int \ln\,(1-ix)\,dx \tag{5b}$$

$$= -i\int \ln\,(1+ix)\,d(ix) + i\int \ln\,(1-ix)\,d(-ix) \tag{5c}$$

$$= -i\int \ln\,(1+ix)\,d(1+ix) + i\int \ln\,(1-ix)\,d(1-ix) \tag{5d}$$

$$= -i\int \ln\,(w_+)\,d(w_+) + i\int \ln\,(w_-)\,d(w_-)\,, \tag{5e}$$

where $w_+ \equiv 1+ix$ and $w_- \equiv 1-ix$.

Then,

$$I = -iw_+(\ln w_+ - 1) + iw_-(\ln w_- - 1) \tag{6a}$$

$$= -i(1+ix)(\ln\,(1+ix) - 1) + i(1-ix)(\ln\,(1-ix) - 1) \tag{6b}$$

$$= i\big[(\ln\,(1-ix) - \ln\,(1+ix))\big] + x\big[(\ln\,(1+ix) + \ln\,(1-ix))\big] - i(-i+x) - (i+x) \tag{6c}$$

$$= i\ln\frac{(1-ix)}{(1+ix)} + x\ln\,(1+x^2) - 2x \tag{6d}$$

$$= i\ln\frac{(1-ix)^2}{(1+x^2)} + x\ln\,(1+x^2) - 2x\,. \tag{6e}$$

Now,

$$\overline{I} = I_0^1 = I(1) - I(0) = I(1)\,. \tag{7}$$

So,

$$\overline{I} = i\ln\frac{(1-i)^2}{2} + \ln 2 - 2 \tag{8a}$$

$$= i\ln\left(\frac{(1-i)}{\sqrt{2}}\right)^2 + \ln 2 - 2 \tag{8b}$$

$$= i\ln\left(e^{-i\pi/4}\right)^2 + \ln 2 - 2 \tag{8c}$$

$$= i\ln e^{-i\pi/2} + \ln 2 - 2 \tag{8d}$$

$$= i(-i\pi/2) + \ln 2 - 2 \tag{8e}$$

$$= \pi/2 + \ln 2 - 2\,. \tag{8f}$$

This is the same answer that WolframAlpha presented.