# Mathematics Diversions 11

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**Abstract**

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube: 'Solving a differential equation'.

The YouTube video is found at:

`https://www.youtube.com/watch?v=tlUlPMaCY4k`

Titled: Solving A Differential Equation | Two Methods

By presenter: SyberMath

## 1 The Problem

The problem is to solve for $y$ as a function of $x$, given the differential equation

$$\left(\frac{dy}{dx}\right)^2 - 1 = x^2 \,. \tag{1}$$

My plan is this:

1) Find the appropriate integral.

2) Integrate using hyperbolic trigonometric functions.

3) Convert the answer to the form given by the presenter.

## 2 The Prerequisites

I don't usually merely reproduce the methods used in a YouTube math video. My presentation of this integration will be a bit different, especially since it uses hyperbolic trig functions and the unipodal algebra.

**A.** Hyperbolic Identities. This will be a brief treatment.

We begin with the fundamental identity: $u^2 = 1$, but $u$ is not a complex number. Then

$$e^{ux} = \cosh x + u \sinh x\,, \tag{2}$$

which can be established by expanding $e^{ux}$ in a power series. On replacing $u$ by $-u$, we get

$$e^{-ux} = \cosh x - u \sinh x\,. \tag{3}$$

Now, since $e^{ux}e^{-ux} = 1$, we get

$$1 = \cosh^2 x - \sinh^2 x\,. \tag{4}$$

Then we have the simple $\left(e^{ux}\right)^2 = e^{2ux}$. After expanding both sides of this equation we get

$$(\cosh x + u \sinh x)^2 = \cosh 2x + u \sinh 2x\,. \tag{5}$$

The complex part is

$$\cosh^2 x + \sinh^2 x = \cosh 2x\,, \tag{6}$$

and the uniplex part is[1]

$$2u \cosh x \sinh x = u \sinh 2x\,. \tag{7}$$

Collecting these, we have

$$\cosh^2 x - \sinh^2 x = 1\,, \tag{8a}$$
$$\cosh^2 x = \tfrac{1}{2}(1 + \cosh 2x)\,, \tag{8b}$$
$$\sinh 2x = 2 \cosh x\, \sinh x\,. \tag{8c}$$

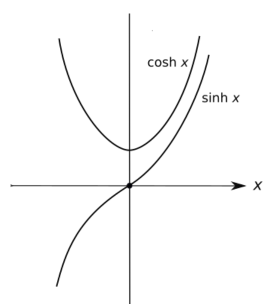


Figure 1. The hyperbolic cosine and sine when $x$ is real valued.

[1]A general unipodal number can be repesented as the ’complex part’ + ’uniplex part’. You can decide for yourself if the uniplex part should contain the factor of $u$ or not. When I first formulated this number system, I thought of it as a Clifford algebra of one dimension over the complex numbers (the ‘scalars’). At that time, I thought of a unipodal number as the sum of the scalar part (which was a complex number) and the vector part (which was a complex number times a vector, the unit vector $u$ in particular).

When we allow the variable $x$ above to be complex-valued, we are able to mix together both circular and hyperbolic numbers and thus functions. That's very convenient for proving formulas.

$$\cosh^2 x - \sinh^2 x = 1\,, \tag{9a}$$

$$\cosh^2 x = \tfrac{1}{2}(1 + \cosh 2x)\,, \tag{9b}$$

$$\sinh 2x = 2\cosh x\, \sinh x\,. \tag{9c}$$

Next, let's add in a couple differential identities:

$$\frac{d}{dx}\cosh x = \sinh x\,, \tag{10a}$$

$$\frac{d}{dx}\sinh x = \cosh x\,. \tag{10b}$$

They can be established by differentiating their power series forms.

**B.** The unipodal algebra. This algebra is formed as the extension of the complex number by the number $u$, where $u^2 = 1$, and $u$ commutes with the complex numbers. The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{11a}$$

$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{11b}$$

$$u_\pm^2 = u_\pm\,, \tag{11c}$$

$$u = u_+ - u_-\,, \tag{11d}$$

$$u_+ u_- = 0\,, \tag{11e}$$

$$u_+ + u_- = 1\,. \tag{11f}$$

You should prove (11c) – (11f). You might also try to prove that the $\cosh x$ is an even function of its argument, and $\sinh x$ is an odd function of its argument. By the way, these two unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses.

**Lemma 1:** Let $f$ be an analytic function of its argument. This means that it can be expressed in terms of a power series. Then

$$f(au_+ + bu_-) = f(a)u_+ + f(b)u_-\,. \tag{12}$$

The proof is straightforward since both $u_+$ and $u_-$ square to be themselves and they mutually annihilate each other.

**Lemma 2:** Let $\gamma$ be real or complex-valued variable. Then

$$e^{u\gamma} = \cosh\gamma + u\sinh\gamma\,. \tag{13}$$

Since the exponential function can be expanded in a power series of its argument, we could prove this lemma by that means, but here I offer an alternative proof:

$$
\begin{aligned}
e^{u\gamma} &= e^{(u_+ - u_-)\gamma} = e^{(\gamma)u_+ + (-\gamma)u_-} \\
&= e^{\gamma}u_+ + e^{-\gamma}u_- \\
&= e^{\gamma}\tfrac{1}{2}(1+u) + e^{-\gamma}\tfrac{1}{2}(1-u) \\
&= \tfrac{1}{2}[e^{\gamma} + e^{-\gamma}] + u\tfrac{1}{2}[e^{\gamma} - e^{-\gamma}] \\
&= \cosh\gamma + u\sinh\gamma\,.
\end{aligned}
$$

## 3 The Solution: Part 1

We quickly arrive at the integral

$$y = \int dy = \int dx\sqrt{x^2+1}\,. \tag{14}$$

From (11a), we can write

$$\cosh^2 x = \sinh^2 x + 1\,. \tag{15}$$

So, let's make the variable substitution

$$x = \sinh\gamma\,. \tag{16}$$

Then

$$dx = \cosh\gamma\, d\gamma\,, \tag{17}$$

$$\sqrt{1+x^2} = \cosh\gamma\,. \tag{18}$$

On substituting into (14), we get

$$y = \int d\gamma\cosh^2\gamma\,. \tag{19}$$

This will integrate easily by applying a few indentities, which we have already established.

$$
\begin{aligned}
y &= \int d\gamma\cosh^2\gamma \\
&= \tfrac{1}{2}\int d\gamma(1+\cosh 2\gamma) \\
&= \tfrac{1}{2}\gamma + \tfrac{1}{2}\int d\gamma\cosh 2\gamma \\
&= \tfrac{1}{2}\gamma + \tfrac{1}{4}\int d(2\gamma)\cosh 2\gamma \\
&= \tfrac{1}{2}\gamma + \tfrac{1}{4}\sinh 2\gamma + c\,,
\end{aligned}
$$

where $c$ is an arbitrary constant. Continuing, we have that

$$\begin{aligned} y &= \tfrac{1}{2}\gamma + \tfrac{1}{4}\sinh 2\gamma + c \\ &= \tfrac{1}{2}\gamma + \tfrac{1}{2}\cosh\gamma\sinh\gamma + c \\ &= \tfrac{1}{2}\sinh^{-1} x + \tfrac{1}{2}x\sqrt{x^2+1} + c\,. \end{aligned} \tag{20}$$

Now, the answer presented is (to within a sign)[2]

$$y = \tfrac{1}{2}\ln\left| x + \sqrt{x^2+1}\right| + \tfrac{1}{2}x\sqrt{x^2+1} + c\,. \tag{21}$$

# 4 The Solution: Part 2

There is a definite preference in the literature to use the logarithm over the hyperbolic trigonometric functions when presenting integrals. No problem. I just need to show that

$$\sinh^{-1} x = \ln\left| x + \sqrt{x^2+1}\right|\,. \tag{22}$$

Okay,

$$x = \sinh\gamma = \frac{e^{u\gamma} - e^{-u\gamma}}{2u}\,. \tag{23}$$

Next, multiply through by $e^{u\gamma}$ and clear of fractions:

$$2xue^{u\gamma} = e^{2u\gamma} - 1\,. \tag{24}$$

Now we let $\mathbb{X} \equiv e^{u\gamma}$ and re-express in standard polynomial form:

$$\mathbb{X}^2 - 2xu\mathbb{X} - 1 = 0\,. \tag{25}$$

Now, the unipodal numbers form a ring but not a division ring, since some of its nonzero numbers do not have inverses. However, the quadratic formula does not employ inverses in its derivation, so we can apply it to (25), to get

$$\mathbb{X}_{\pm} = \frac{2xu \pm \sqrt{4x^2+4}}{2} = xu \pm \sqrt{x^2+1}\,. \tag{26}$$

However, since $\sqrt{x^2+1}$ represents the $\cosh x$, which is always positive, we should only use the positive form of the root, thus,

$$e^{u\gamma} = \mathbb{X}_{+} = xu + \sqrt{x^2+1}\,. \tag{27}$$

Then,

$$\begin{aligned} \ln[e^{u\gamma}] = u\gamma &= \ln[\sqrt{x^2+1}(u_+ + u_-) + x(u_+ - u_-)] \\ &= \ln[(\sqrt{x^2+1} + x)u_+ + (\sqrt{x^2+1} - x)u_-] \\ &= \ln[\sqrt{x^2+1} + x]u_+ + \ln(\sqrt{x^2+1} - x)u_- \\ &= \ln[\sqrt{x^2+1} + x]\tfrac{1}{2}(1+u) + \ln(\sqrt{x^2+1} - x)\tfrac{1}{2}(1-u)\,, \end{aligned} \tag{28}$$

[2]The answer I got seems to agree with my table of integrals.

where I used the common trick, to employ the identity for any number $K$:

$$K = K(u_+ + u_-) = Ku_+ + Ku_- \,, \tag{29}$$

which uses (11f).

On further expanding and simplifying, we have that

$$\begin{aligned} 2u\gamma &= u \ln \left| \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}-x} \right| \\ &= u \ln \left| (\sqrt{x^2+1}+x)^2 \right| \\ &= 2u \ln \left| \sqrt{x^2+1}+x) \right| \,, \end{aligned} \tag{30}$$

where we used that

$$(\sqrt{x^2+1}+x)(\sqrt{x^2+1}-x) = 1 \,. \tag{31}$$

Finally, we get that

$$\sinh^{-1} x = \gamma = \ln \left| \sqrt{x^2+1}+x \right| \,. \tag{32}$$

And this establishes Eq. (21).