Mathematics Diversions 12

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Abstract

Here we use hyperbolic trigonometric identities to assist in solving the problem, which is given to us on YouTube: ‘Solving a Quick and Easy Functional Equation’.

The YouTube video is found at:

`https://www.youtube.com/watch?v=zLU5Cagv-Tc`

Titled: Solving a Quick and Easy Functional Equation

By presenter: SyberMath

1 The Problem

The problem is to solve for $f(x)$ given the relation

$$f(x+\sqrt{x^2+1}) = \frac{x}{x+1}\,. \tag{1}$$

My plan is this:

1) Use some of the hyperbolic identities from the last math diversion paper.

2 The Prerequisites

I don’t usually merely reproduce the methods used in a YouTube math video. My presentation of solution will be a bit different, especially since it uses hyperbolic trig functions (but without the unipodal algebra this time).

A. Hyperbolic Identities. This will be a brief treatment.

$$e^x = \cosh x + \sinh x\,, \tag{2}$$
$$e^{-x} = \cosh x - \sinh x\,. \tag{3}$$

On combinging these, we get

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}), \tag{4a}$$

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}). \tag{4b}$$

$$\cosh^2 x - \sinh^2 x = 1, \tag{4c}$$

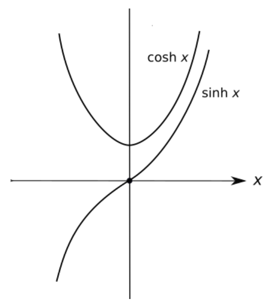

Figure 1. The hyperbolic cosine and sine when x is real valued.

Now, we let $x \to \sinh y$ and thus $\sqrt{x^2+1} \to \cosh y$.

$$e^y = \cosh y + \sinh y = x + \sqrt{x^2+1}. \tag{5}$$

So, on substituting this into (1), get

$$f(e^y) = \frac{\sinh y}{\sinh y + 1}. \tag{6}$$

We're almost finished. All we need to do now is to replace $\sinh y$ by its exponential equivalence.

$$f(e^y) = \frac{\frac{1}{2}(e^y - e^{-y})}{\frac{1}{2}(e^y - e^{-y}) + 1} = \frac{e^{2y} - 1}{e^{2y} + 2e^y - 1}. \tag{7}$$

Next, we replace e^y by t, to get

$$f(t) = \frac{t^2 - 1}{t^2 + 2t - 1}. \tag{8}$$

And, of course, we can recast it in terms of x as

$$f(x) = \frac{x^2 - 1}{x^2 + 2x - 1}. \tag{9}$$