Math Diversion Problem 123

P. Reany

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> People often overlook the obvious.
> — Doctor Who

The YouTube video is found at:

Source: https://www.youtube.com/watch?v=ggDoEEcdBp0
Title: Crack This Olympiad Exponential Equation
 Presenter: GT Academix

1 The Problem

Given the relation

$$x^{x^{x^3}} = 3\,, \tag{1}$$

find the real values of x.

2 The Solution

The presenter seemed to have an insightful guess into the nature of the problem that I lacked when I attempted this problem. (When we do need to guess, guess simple!) My solution began by making the variable substitution

$$x = 3^y\,. \tag{2}$$

This is part of the grace one has when working with functions of a real variable. Anyway, after making this substitution into (1), we have

$$(3^y)^{(3^y)^{(3^y)^3}} = 3\,, \tag{3}$$

which becomes

$$(3^y)^{(3^y)^{(3^{3y})}} = (3^y)^{(3^{y3^{3y}})} = 3^{y3^{y3^{3y}}} = 3^1\,. \tag{4}$$

From this, we demand equal exponents:

$$y3^{y3^{3y}} = 1\,. \tag{5}$$

At this point, the insightful guess to the solution seems to me a bit easier here than at (1), from which I get $y = 1/3$, and thus $x = 3^{1/3} = \sqrt[3]{3}$.