

Math Diversion Problem 124

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Keep an open mind. That's the secret.
— Doctor Who

Heuristic note: When it comes time to guess, start by guessing simple.
— The Author

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=nJwQGTPIyUM>
Title: A Nice Math Olympiad Exponential Equation $X^x = 16$
Presenter: MrMath

1 The Problem

Given the relation

$$x^{x^2} = 16, \tag{1}$$

find the real values of x .

2 The Solution

The value 16 on the RHS is not a big number for the problem we have. We could (and maybe should) just start trying low integer values for x . Doing that, we quickly find that $x = 2$ is a solution, but then $x = -2$ is also a solution.

On the other hand, a more systematic approach could begin by making the variable substitution

$$x = 2^y \quad (\text{for } x > 0), \tag{2}$$

which would give us

$$2^{y2^{2y}} = 2^4 = 2^{2^2}. \tag{3}$$

But on demanding equal exponents and using some algebra, yields:

$$y2^{2y-2} = 1. \tag{4}$$

At this point, a reasonable guess is $y = 1$, making $x = 2$. But that's the solution for positive x . Clearly, we also have $x = -2$ as a solution.