Math Diversion Problem 127

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A clue is anything that doesn't happen
the way it oughtta happen.
— Harry Orwell, TV
show *Harry O*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Uqk2MS5Lcsk
Title: Math Olympiad | Algebra Problem
 Presenter: MathsFocus
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1 The Problem

Given the relation

$$\frac{2^{33}+2^{22}+2^{11}}{2^{33}-1}\,, \tag{1}$$

find its value.

2 The Solution

If you don't already know about the geometric series, this is a good time to learn. Consider the series

$$1+r+r^2+r^3+\cdots+r^n\,, \tag{2}$$

where r is the ratio one any term to its previous term. This series has the closed form value

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1} \quad (r\neq 1)\,. \tag{3}$$

This finite series will always converge. If it's infinite, it will always converge over complex or real r if $|\,r\,|<1$.

So, let's return to (1):

$$\frac{2^{33}+2^{22}+2^{11}}{2^{33}-1} = \frac{2^{11}}{2^{33}-1}[(2^{11})^2+2^{11}+1]\,. \tag{4}$$

Here $r = 2^{11}$ and $n = 2$,

$$(2^{11})^2 + 2^{11} + 1 = \frac{(2^{11})^3 - 1}{2^{11} - 1} = \frac{2^{33} - 1}{2^{11} - 1}\,. \tag{5}$$

On plugging this value into (4), we have that

$$\frac{2^{33} + 2^{22} + 2^{11}}{2^{33} - 1} = \frac{2^{11}}{2^{33} - 1}\frac{2^{33} - 1}{2^{11} - 1} = \frac{2^{11}}{2^{11} - 1} = \frac{2048}{2047}\,. \tag{6}$$