

Math Diversion Problem 128

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Mathematics knows no races or geographic boundaries;
for mathematics, the cultural world is one country.

— David Hilbert

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=VJmVcpfcy3Q>
Title: A Tricky Math Olympiad challenge
Presenter: MathsFocus

1 The Problem

Given the relation

$$x^{x^6} = \sqrt{2}^{\sqrt{2}}, \quad (1)$$

find the values of x .

2 The Solution

I'm going to use my standard 'algebraic' approach before attempting to guess the right answer. So, let

$$x = \sqrt{2}^\alpha \quad (x > 0). \quad (2)$$

On plugging this value into (1), we have that

$$(\sqrt{2}^\alpha)^{(\sqrt{2}^\alpha)^6} = (\sqrt{2}^\alpha)^{(\sqrt{2}^{6\alpha})} = \sqrt{2}^{\alpha(\sqrt{2}^{6\alpha})} = \sqrt{2}^{\sqrt{2}}. \quad (3)$$

On setting exponents equal, we have that

$$\alpha(\sqrt{2}^{6\alpha}) = \sqrt{2}, \quad (4)$$

which can be simplified a bit to

$$\alpha \sqrt{2}^{6\alpha-1} = 1. \quad (5)$$

With some trial and error, I get $\alpha = 1/2$. Thus $x = \sqrt[4]{2}$.