Math Diversion Problem 132

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

After a time, you may find that having is not so
pleasing a thing after all as wanting. It is not
logical, but is often true.
— Spock

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=oMhDv9tydW0
Title: Can you Solve a tricky Entrance Exam from
       Cambridge University ?
 Presenter: Super Academy
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1 The Problem

Given the relation

$$\frac{16}{x} - \frac{8}{x^2} + \frac{4}{x^3} - \frac{2}{x^4} + \frac{4}{x^5} = 32\,, \tag{1}$$

find the values of x.

2 The Solution

If you don't already know about the geometric series, this is a good time to learn. Consider the series

$$1 + r + r^2 + r^3 + \cdots + r^n\,, \tag{2}$$

where r is the ratio one any term to its previous term. This series has the closed form value

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1} \quad (r \neq 1)\,. \tag{3}$$

This finite series will always converge. If it's infinite, it will always converge over complex or real r if $|\,r\,| < 1$.

Now, my first instinct is to rewrite the given equation to this

$$32x^5 - 16x^4 + 8x^3 - 4x^2 + 2x - 1 = 0\,. \tag{4}$$

This is a geomeric series where $r = -2x$. Applying this to the formula in (3), we have that

$$\frac{(-2x)^6 - 1}{(-2x) - 1} = 0\,, \tag{5}$$

where we insist that $-2x \neq 1$. Okay, so what values do we get for x from the following equation?

$$(-2x)^6 = 1\,, \tag{6}$$

First, according to the above restriction, we can't allow $x = -\frac{1}{2}$. But we can allow $x = \frac{1}{2}$. We can also allow values of the form

$$x = -\tfrac{1}{2} e^{2i\pi\alpha}\,, \tag{7}$$

where $(e^{2i\pi\alpha})^6 = e^{2i\pi}$ or an integer power of this. Suitable α's are $\frac{1}{6}$ and $\frac{1}{3}$. Then

$$x = \pm\tfrac{1}{2} e^{i\pi/3} = \pm\tfrac{1}{4}(-1 + i\sqrt{3})\,, \quad x = \pm\tfrac{1}{2} e^{2i\pi/3} = \pm\tfrac{1}{4}(-1 - i\sqrt{3})\,. \tag{8}$$

And that makes a total of five solutions.