Math Diversion Problem 135

P. Reany

November 10, 2024

Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

Our greatest weakness lies in giving up. The most certain way to succeed is always to try just one more time.
—Thomas Edison

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=67234_o6z-4
Title: Germany | A nice Logarithmic Math Olympiad Problem
 Presenter: Super Academy
```

1 The Problem

Given the relation

$$x^{\log 27} + 9^{\log x} = 36\,, \tag{1}$$

find the values of x over the real numbers. By the way, I interpret the logarithm as being in base 10.

2 The Solution

The first thing I did was to introduce a change of variable. Let

$$x = 10^{\alpha}\,, \tag{2}$$

so that on substituting, we get

$$(10^{\alpha})^{\log 27} + 9^{\log 10^{\alpha}} = 36\,, \tag{3}$$

and this simplifies to

$$27^{\alpha} + 9^{\alpha} = 36\,. \tag{4}$$

Let's introduce another variable change:

$$y = 3^{\alpha}\,, \tag{5}$$

then, (4) becomes

$$y^3 + y^2 - 36 = 0\,. \tag{6}$$

When I tried some small integers for y, I found that $y = 3$ is one solution.

Then, on factoring (6), we get

$$(y-3)(y^2 + 4y + 12) = 0\,. \tag{7}$$

The roots to the quadratic are complex, so I will ignore them.

Thus, for $y = 3$, we get,

$$\alpha = \frac{\log y}{\log 3} = \frac{\log 3}{\log 3} = 1\,. \tag{8}$$

Hence,

$$x = 10^1 = 10\,. \tag{9}$$