Math Diversions, Problem 14

P. Reany

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1 Problem

This problem is found on the YouTube channel **Spencer's Academy**, from April 28, 2024. My solution here is a little different from that given by the presenter at:

`https://www.youtube.com/watch?v=k2QONlmEab0`

Titled: A very nice olympiad question | How to solve

$$(4+\sqrt{15})^x + (4-\sqrt{15})^x = 62\,. \tag{1}$$

(Note: The above equation is in corrected form.)

Statement of the problem:

Solve (1) for real values of x.

Solution: (Not restricted to Olympiad rules.)

First, we'll introduce two new variables a_+ and a_- as follows:

$$a_+ = 4+\sqrt{5}\,, \tag{2a}$$

$$a_- = 4-\sqrt{5}\,. \tag{2b}$$

Then Equation (1) becomes

$$a_+^x + a_-^x = 62\,. \tag{3}$$

As is common practice, we should calculate the sum and product of these new variables:

$$a_+ + a_- = 8\,, \tag{4a}$$

$$a_+ a_- = 1\,. \tag{4b}$$

Next, we introduce the new variable y to form an equation complementary to (1):

$$(4+\sqrt{15})^x - (4-\sqrt{15})^x \equiv y\,, \tag{5}$$

which we'll then translate into our new variables:

$$a_+^x - a_-^x = y\,. \tag{6}$$

By taking the sum and difference of Eq. (3) and (6), we get

$$2a_+^x = 62 + y\,, \tag{7a}$$
$$2a_-^x = 62 - y\,. \tag{7b}$$

And on taking the product of these last two equations and using (4b), gives us

$$y^2 = 62^2 - 4\,. \tag{8}$$

From this last equation and (7a), we get

$$a_+^x = \tfrac{1}{2}(62 + y) = \tfrac{1}{2}(62 \pm \sqrt{62^2 - 4})\,. \tag{9}$$

Now, I'm not restricting myself to Olympiad-only techniques, so I will take the logarithm on both sides and then put that result into the Wolframalpha.com equation solver:

$$x = \frac{\log[\frac{1}{2}(62 \pm \sqrt{62^2 - 4})]}{\log(4 + \sqrt{15})}\,. \tag{10}$$

When I try:

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Solve for x,  x = \log[(1/2)(62+\sqrt{62^2-4})]/\log(4 + \sqrt{15})
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I get back 2.0.

Trying $x = 2$ in (1), I confirm that it's a solution.

When I try:

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Solve for x,  x = \log[(1/2)(62-\sqrt{62^2-4})]/\log(4 + \sqrt{15})
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I get back -2.0. But because of (4b), this must be a solution as well:

$$a_+^{-2} + a_-^{-2} = a_-^2 + a_+^2 = 62\,. \tag{11}$$