Math Diversion Problem 141

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

First things first...But not necessarily in that order.
— Doctor Who

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Gz6VLgvFLHs
Title: A nice Math Olympiad Simplification Problem
 Presenter: Super Academy
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1 The Problem

Given the relation

$$3^{3x-3x^2} = x^2 - x\,, \tag{1}$$

find the values of x over the real numbers.

2 The Solution

We begin by setting

$$y = x^2 - x\,. \tag{2}$$

Then

$$3^{3(-y)} = y\,, \tag{3}$$

which can be rewritten as

$$y27^y = 1\,. \tag{4}$$

Next, we introduce the variable transformation z to set up the Lambert W function:

$$27^y = e^z \quad \text{and} \quad y = z/(\ln 27)\,, \tag{5}$$

thus (4) becomes

$$ze^z = \ln 27\,. \tag{6}$$

So,

$$z = W(ze^z) = W(\ln 27)\,. \tag{7}$$

Going back to y, we get

$$y = \frac{z}{\ln 27} = \frac{W(\ln 27)}{\ln 27} \equiv \beta\,. \tag{8}$$

Now we can form a quadratic in x from (2):

$$x^2 - x - \beta = 0\,. \tag{9}$$

Now, WolframAlpha claims that $\beta = 1/3$, so I'll go with that.

$$x^2 - x - 1/3 = 0\,. \tag{10}$$

The roots to this are

$$x = \frac{1 \pm \sqrt{7/3}}{2}\,. \tag{11}$$