# Mathematics Diversions 15

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The YouTube short video is found at:

`https://www.youtube.com/shorts/GkDVLurV3gY`

Titled: A Nice Algebraic Expansion Problem

By presenter: ?

## 1 The Problem

Given the relation

$$a + \frac{1}{a} = 7\,, \tag{1}$$

where $a$ is a positive real number, find

$$\sqrt{a} + \frac{1}{\sqrt{a}}\,, \tag{2}$$

and

$$\sqrt{a} - \frac{1}{\sqrt{a}}\,. \tag{3}$$

Now, the problem only asks us to find the first expression (2), but I see no reason not to include the second for very little extra effort! The thing is, the solution is going to be made 'simpler' by use of the unipodal algebra, which is explained below.

## 2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number $u$, where $u^2 = 1$, and $u$ commutes with the complex numbers. The number $u$ is said to be 'unipotent'.

The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{4a}$$
$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{4b}$$
$$u_\pm^2 = u_\pm\,, \tag{4c}$$
$$u = u_+ - u_-\,, \tag{4d}$$
$$u_+ u_- = 0\,, \tag{4e}$$
$$u_+ + u_- = 1\,, \tag{4f}$$
$$u u_+ = u_+\,, \tag{4g}$$
$$u u_- = -u_-\,. \tag{4h}$$

You should prove (4c) – (4h). By the way, these two unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots. It reminds me of when I was a kid, and other kids would fold a piece of paper in such a way that they could, with two fingers of each hand, open and close the folded paper in two different ways. The practice of this folding is called origami. (Some call the result of that folding the 'Fortune Teller' fold.) But I think of this construction as an analogy: The paper represents a unipode: Open it one way to see the number in the standard basis, and open it the other way to see it in the idempotent basis.

## 3 The Solution

We define the unipode $X$ by

$$X \equiv \sqrt{a} + \frac{1}{\sqrt{a}}\,u\,, \tag{5}$$

which is in the standard basis. Now, we'll square it:

$$X^2 = \left(a + \frac{1}{a}\right) + 2u = 7 + 2u\,, \tag{6}$$

where we used (1). Next, we expand $X^2$ in the idempotent basis.

$$X^2 = 9u_+ + 5u_-\,. \tag{7}$$

Then, we take the square root across this last equation (there will be four of them, two for each idempotent):

$$X = \pm 3u_+ \pm \sqrt{5}u_- \,. \tag{8}$$

Now, if we convert $X$ in (5) to the idempotent basis, we get

$$X = \left(\sqrt{a} + \frac{1}{\sqrt{a}}\right) u_+ + \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right) u_- \,. \tag{9}$$

Comparing the components of these last two equations we get

$$\sqrt{a} + \frac{1}{\sqrt{a}} = \pm 3 \,, \tag{10a}$$

$$\sqrt{a} - \frac{1}{\sqrt{a}} = \pm\sqrt{5} \,. \tag{10b}$$

## 4 The Conclusion

Is this solution really ‘simpler’ than not using the unipodal algebra? I would say that if the reader is not well familiar with the unipodal algebra, then this solution might not seem simpler. But if the reader is well familiar with the unipodal algebra, then this solution might seem simpler. I regard this problem and its unipodal solution as a sort of training problem for similar, but more involved, problems. There are classes of problems that seem to lend themselves well to using the unipodal algebra, but this isn’t true of all algebraic problems.

Perhaps one of the hardest things to do when using the unipodal algebra to help solve a problem that started off in the rational, real, or complex numbers, is how to pick that first unipodal number to get the solution started. You’re going to have to gin up that unipode out of thin air, so to speak. But to me, that’s part of the fun of it!

My feeling is that if I don’t practice using the unipodal algebra on ‘simple’ problems, then I will never develop the discernment to recognize when I should or should not try to use the unipodal algebra on harder problems. Beyond those concerns, the unipodal algebra is a commutative ‘playground’ algebra for learning about certain important features of noncommutative rings, such as ideals and complementary idempotents, which one is likely to eventually encounter in geometric algebra, Clifford algebra, and the Pauli algebra.