Math Diversion Problem 150

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Mental toughness is essential to success.
— Vince Lombardi

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=VayJZrGnbLI
Title: Russisa | A Nice Algebra Problem
 Presenter: SALogic
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1 The Problem

Given the relation

$$\sqrt[3]{x} + \sqrt{x} = \frac{4}{27}\,, \tag{1}$$

find the values of x over the real numbers.

2 The Solution

I like the idea of clearing of fractions, and that includes exponents, when possible. Let

$$x = u^6\,. \tag{2}$$

Then (1) becomes

$$u^2 + u^3 = \frac{4}{27} \tag{3}$$

Let's clear some more fractions and change the order,

$$(3u)^3 + 3(3u)^2 - 4 = 0\,. \tag{4}$$

One more variable change. Let

$$w = 3u \quad \text{then} \quad u^2 = \frac{w^2}{9}\,. \tag{5}$$

Then (4) becomes

$$w^3 + 3w^2 - 4 = 0\,. \tag{6}$$

By inspection, we can find the root $w = 1$. So, on dividing (6) by the factor $(w - 1)$, we get

$$w^2 + 4w + 4 = 0\,, \tag{7}$$

which has $w = -2$ as a double root. Okay, the roots for w go like this:

$$w = \{1, -2, -2\}\,. \tag{8}$$

The roots for u go like this:

$$u = \{\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}\}\,. \tag{9}$$

The roots for $x = u^6$ go like this:

$$x = \{\frac{1}{3^6}, \left(\frac{-2}{3}\right)^6, \left(\frac{-2}{3}\right)^6\}\,. \tag{10}$$

However, WolframAlpha admits only $x = \dfrac{1}{3^6} = \dfrac{1}{729}$ as a root.