# Math Diversion Problem 156

P. Reany

November 16, 2024

The secret to perseverence is to just keep doing it.
— The Author

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=U5Svh3S7Ziw
Title: A Nice Math Olympiad Algebra Problem
 Presenter: Master T Maths Class
```

## 1 The Problem

Given the relation

$$x^{\sqrt{x}} = 10\,, \tag{1}$$

find the values of $x$ over the real numbers.

## 2 The Solution

I like to approach this kind of problem in a standard way. So, let

$$x = 10^{\alpha}\,. \tag{2}$$

Then (1) becomes

$$(10^{\alpha})^{(10^{\alpha})^{1/2}} = 10\,, \tag{3}$$

which simplifies to

$$10^{\alpha 10^{\alpha/2}} = 10^1\,. \tag{4}$$

On equating exponents, we have that

$$\alpha 10^{\alpha/2} = 1\,. \tag{5}$$

So, how does one approach this equation to solve for $\alpha$? My method is first to look for an integer solution and then a rational solution. If that fails, one can always move things in the direction of Lambert's $W$ function,[1] which I'll do now. Let's divide through by 2:[2]

$$\tfrac{1}{2}\alpha 10^{\alpha/2} = \tfrac{1}{2}\,. \tag{6}$$

Let $\beta = \alpha/2$, then we get

$$\beta 10^{\beta} = \tfrac{1}{2}\,. \tag{7}$$

Okay, we're getting closer. Next, introduce $y$ so that it satifies the relation

$$e^{y} = 10^{\beta}\,, \tag{8}$$

from which we get that

$$\beta = \frac{y}{\ln 10}\,. \tag{9}$$

Next, we substitute these relations into (7):

$$\frac{y}{\ln 10}e^{y} = \tfrac{1}{2}\,. \tag{10}$$

Therefore,

$$y = W(ye^{y}) = W\Big(\frac{\ln 10}{2}\Big)\,. \tag{11}$$

Going back to (9), we have that

$$\beta = \frac{W(\frac{1}{2}\ln 10)}{\ln 10}\,. \tag{12}$$

So,

$$\alpha = 2\frac{W(\frac{1}{2}\ln 10)}{\ln 10} = \frac{W(\frac{1}{2}\ln 10)}{\frac{1}{2}\ln 10}\,. \tag{13}$$

Therefore,

$$x = 10^{\frac{W(\frac{1}{2}\ln 10)}{\frac{1}{2}\ln 10}}\,. \tag{14}$$

Or, you can do something like this:

$$x = 10^{\alpha} = 10^{2\beta}\,, \tag{15}$$

so,

$$\sqrt{x} = 10^{\beta} = e^{y} = e^{W(\frac{1}{2}\ln 10)}\,. \tag{16}$$

Hence

$$x = e^{2W(\frac{1}{2}\ln 10)}\,. \tag{17}$$

[1]The Lambert $W$ function has the remarkable property that (under certain domain restrictions) $W(ze^{z}) = z$.

[2]By the way, make variable substitutions as often as you think necessary.