Math Diversion Problem 158

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> There can be very little of present-day science and
> technology that is not dependent on complex
> numbers in one way or another.
> — Keith Devlin

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=3a0EF9j1jIk
Title: I Couldn't Solve This Equation - WA did! | 428
 Presenter: aplusbi
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1 The Problem

Given the relation

$$\overline{z} + z\,|z| = 24 - 12i\,, \tag{1}$$

find the values of z over the complex numbers.

2 Basics of Complex Numbers

Typically, we find a generic complex number denoted by the letter z, but one is free to choose other letters, as well. So, if z is a complex number, in general it has both real and imaginary parts:

$$z = a + bi\,, \tag{2}$$

where a, b are real components of basis vectors $1, i$. But they are also expressed as, respectively, the 'real' and 'imaginary' components of z.

Complex conjugation of complex number z is an operation that leaves real numbers alone but replaces the unit imaginary i with its negative, i.e., $-i$. The symbols most often used to represent complex conjugation are the $*$ and the overbar. I'll usually use the overbar. Thus, the complex conjugate of z in (2) is

$$\overline{z} = a - bi\,. \tag{3}$$

Obviously, the complex conjugation of a pure real number has no effect.

A funny thing happens when we multiply a complex number by its conjugate:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2\,. \tag{4}$$

So, $z\overline{z}$ is zero if and only if $z = 0$, otherwise, it's a positive real number.

Another funny thing happens when we add a complex number and its conjugate: we also get a real number. Let's see.

$$z + \overline{z} = (a + bi) + (a - bi) = 2a\,. \tag{5}$$

Why do we care about this? Because sometimes we need to map complex numbers into the real numbers to get information on the complex numbers. This problem will show you that.

I'm not going to prove this here, but every complex number can be expressed in exponential (or polar) form:

$$z = a + bi = \sqrt{a^2 + b^2}e^{i\theta} = (z\overline{z})^{1/2}e^{i\theta} = r\,e^{i\theta}\,, \tag{6}$$

where we can think of r as the length of the complex numbers z or $\overline{z}$.

$$r \equiv (z\overline{z})^{1/2} \quad \text{or} \quad r^2 = z\overline{z} = |\,z\,|^2\,. \tag{7}$$

So, it will be good to know all this stuff in this section before you attempt to follow my solutions to these complex variables problems.

By the way, the complex numbers are what's called a *field*, so they can be added, subtracted, multiplied, and divided by each other (except you can't divide by zero, as usual). And, therefore, you can apply the quadratic formula to them! (Yay!)

Lemma 1: If a complex number z is equal to its own conjugate $z = \overline{z}$, it's real.

Lemma 2: If a complex number z is complex conjugated twice then there's no change: $\overline{\overline{z}} = z$.

Lemma 3: The complex conjugated of a product or a sum is the product or sum of the complex conjugates: $\overline{z_1 z_2} = \overline{z}_1\overline{z}_2$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$.

3 The Solution

I'd like to write down the (complex) conjugate of (1) because we'll need it:

$$z + \overline{z}\,|\,z\,| = 24 + 12i\,. \tag{8}$$

Before actually solving the problem, I'd also prefer to recaste the equations to the following simpler forms:

$$\overline{z} + zr = \overline{z}_0\,, \tag{9a}$$

$$z + \overline{z}r = z_0\,, \tag{9b}$$

where

$$z_0 = 24 + 12i \quad \text{and} \quad \overline{z}_0 = 24 - 12i \quad \text{and} \quad z_0\overline{z}_0 = 720\,. \tag{10}$$

Now, since $z\overline{z} = r^2$, we can rewrite (9b) into the form (by eliminating $\overline{z}$)

$$z + \frac{r^3}{z} = z_0\,. \tag{11}$$

Since we already know z_0, we can solve (11) for z as soon as we know the value of r. So, let's start the process of solving for r.

When we multiply (9a) and (9b) together, we get

$$(\overline{z} + zr)(z + \overline{z}r) = \overline{z}_0 z_0 = r_0^2\,, \tag{12}$$

which becomes

$$r^2 + r(z^2 + \overline{z}^2) + r^4 = 720\,. \tag{13}$$

So, the hang-up here is to find an expression in r for $z^2 + \overline{z}^2$. Now,

$$(z + \overline{z})^2 = z^2 + \overline{z}^2 + 2z\overline{z} = z^2 + \overline{z}^2 + 2r^2\,, \tag{14}$$

which can be rewritten as

$$z^2 + \overline{z}^2 = (z + \overline{z})^2 - 2r^2\,, \tag{15}$$

which could be a little closer to what we want. Okay, from (9a) and (9b) we have that

$$z + \overline{z} = (\overline{z}_0 - zr) + (z_0 - \overline{z}r) = (z_0 + \overline{z}_0) - r(z + \overline{z})\,, \tag{16}$$

where I'm trying to convert to the numbers a and a_0. But this can be rewritten as

$$2a = 2a_0 - r2a\,. \tag{17}$$

On solving this for a, we get

$$a = \frac{a_0}{1 + r}\,. \tag{18}$$

So,

$$z + \overline{z} = 2a = \frac{2a_0}{1 + r}\,. \tag{19}$$

Then, first, (15) can be rewritten as

$$z^2 + \overline{z}^2 = \left(\frac{2a_0}{1 + r}\right)^2 - 2r^2\,. \tag{20}$$

Therefore, (13) can be rewritten as

$$r^2 + r\left[\left(\frac{2a_0}{1 + r}\right)^2 - 2r^2\right] + r^4 = 720\,, \tag{21}$$

or

$$r^4 + r^2 + r\left[\frac{4a_0^2}{(1+r)^2} - 2r^2\right] = 720\,. \tag{22}$$

When I gave this last equation to WolframAlpha to solve for r, it sent back

$$r = 5\,, \tag{23}$$

as the only real, positive solution (as r must be). And when I put that value in for r in (11), WolframAlpha sent back for z:

$$z = 4 - 3i\,, \tag{24}$$

which is at least consistent with a length of $r = 5$.

4 Conclusion

This solution apparently worked with a minimal amout of use of the components a, b of which z is composed.[1] But, the proof seems to me, subjectively, at least a third too long, leading me to think that I overlooked a better demonstration.

[1] The minimal use of the components of the complex numbers involved is one of my goals.