Mathematics Diversions 16

P. Reany

August 19, 2024

Abstract

This is such an interesting problem to me that I decided to presented it, even though my solution is not much different from the Presenter's.

The YouTube video is found at:

`https://www.youtube.com/watch?v=A-SFQ0s4qQw`

Titled: This Oxford Integral Question STUMPED Students

By presenter: dy d Oscar

1 The Problem

Given the relation

$$6 + f(x) = 2f(-x) + 3x^2 \int_{-1}^{1} f(t)\, dt \,, \tag{1}$$

we are asked to find the value of the integral

$$I \equiv \int_{-1}^{1} f(t)\, dt \,, \tag{2}$$

Hint: We don't know that $f(x)$ is differentiable, but we do know that it is integrable.

2 The Solution

We begin by rewriting (1) as

$$6 + f(x) = 2f(-x) + 3Ix^2 \,. \tag{3}$$

Next, let's test the evenness or oddness of $f(x)$ by replacing x by $-x$:

$$6 + f(-x) = 2f(x) + 3I(-x)^2 \,. \tag{4}$$

On subtracting this last equation from its predecessor, we get

$$f(x) - f(-x) = 2f(-x) - 2f(x) = -2(f(x) - f(-x))\,. \tag{5}$$

But for this to be true

$$f(x) - f(-x) = 0\,, \tag{6}$$

and thus $f(x)$ is an even function.

Now, let's integrate across (3), using the provided limits.

$$\int_{-1}^{1} 6\,dx + \int_{-1}^{1} f(x)\,dx = 2\int_{-1}^{1} f(-x)\,dx + I\int_{-1}^{1} 3x^2\,dx\,, \tag{7}$$

where $f(-x)$ is being treated as $f(x)$. Then

$$12 + I = 2I + Ix^3\Big|_{-1}^{1}\,, \tag{8}$$

or

$$12 + I = 2I + 2I\,. \tag{9}$$

And this yields

$$I = 4\,. \tag{10}$$