Math Diversion Problem 161

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It is clear that the chief end of mathematical study
must be to make the students think.
— John Wesley Young
(So think!)

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=V1ye2oXNp-Y
Title: Russian | Can you solve this ?
 Presenter: Master T Maths Class
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1 The Problem

Given the relation

$$x^{-x^x} = 2^{\sqrt{2}}\,, \tag{1}$$

find the values of x over the real numbers.

2 The Solution

I like to approach this kind of problem in a standard way, by making a variable substitution. But before I do that, I want to change the look of the problem a little. Let's begin by raising both sides to the power of the square root of 2:

$$\left(x^{-x^x}\right)^{\sqrt{2}} = \left(2^{\sqrt{2}}\right)^{\sqrt{2}}\,, \tag{2}$$

which gives us

$$x^{-\sqrt{2}x^x} = 2^2\,. \tag{3}$$

Next, we again simplify by taking the square root of both sides:

$$x^{-\frac{\sqrt{2}}{2}x^x} = 2\,. \tag{4}$$

Now, for my standard form of substitution:

$$x = 2^{\alpha}\,. \tag{5}$$

When we put this into (4), we get that

$$(2^{\alpha})^{-\frac{\sqrt{2}}{2}(2^{\alpha})^{(2^{\alpha})}} = 2\,. \tag{6}$$

This simplifies to

$$2^{-\alpha\frac{\sqrt{2}}{2}(2^{\alpha(2^{\alpha})})} = 2\,. \tag{7}$$

On equating exponents, we have that

$$-\alpha\frac{\sqrt{2}}{2}(2^{\alpha(2^{\alpha})}) = 1\,. \tag{8}$$

So, how does one approach this equation to solve for α? My method is first to look for an integer solution and then a rational solution. If that fails, one can always move things in the direction of Lambert's W function.

But let's begin easy for starters. I tried $\alpha = -1$, but that didn't work. Then I tried $\alpha = -2$, and that did work. You can establish that for yourself, if you like.

Anyway, that means that

$$x = 2^{-2} = \frac{1}{4}\,. \tag{9}$$