Math Diversion Problem 162

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A clue is anything that doesn't happen
the way it oughtta happen.
— Harry Orwell, TV
show *Harry O*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=iEWZXZ8XLew
Title: A Sum of Powers | Problem 421
 Presenter: aplusbi
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1 The Problem

Given the relation

$$\phi = \left(\frac{1-i}{1+i}\right)^2 + \left(\frac{1-i}{1+i}\right)^3 + \left(\frac{1-i}{1+i}\right)^4 , \tag{1}$$

find the values of ϕ over the complex numbers.

2 The Solution

There are some things we need to know up front.[1]

$$e^{i\pi/4} = \frac{1+i}{\sqrt{2}} \quad \text{and} \quad e^{-i\pi/4} = \frac{1-i}{\sqrt{2}} \,. \tag{2}$$

Now watch

$$\frac{e^{i\pi/4}}{e^{-i\pi/4}} = \frac{1-i}{1+i} , \tag{3}$$

but

$$\frac{e^{i\pi/4}}{e^{-i\pi/4}} = e^{i\pi/2} = i \,. \tag{4}$$

[1] What I'm about to show are some needful things to know in complex number theory, even if you don't need them for this problem.

Hence,

$$\frac{1-i}{1+i} = i\,, \tag{5}$$

and, yes, we could have gotten this other ways. Now,

$$\begin{aligned}
\phi &= \left(\frac{1-i}{1+i}\right)^2 + \left(\frac{1-i}{1+i}\right)^3 + \left(\frac{1-i}{1+i}\right)^4 &\text{(6a)}\\
&= (i)^2 + (i)^3 + (i)^4 &\text{(6b)}\\
&= -1 + -i + 1 = -i\,. &\text{(6c)}
\end{aligned}$$