Olympiad Problem 6: Math Diversion 17

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August 20, 2024

Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

The YouTube video is found at:

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https://www.youtube.com/watch?v=-70c48Fl-x4
Titled: Evaluating x sqrt(x)+y sqrt(y)_Math is math
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By presenter: Math is math

1 The Problem

Given the relations

$$x + y = 19\,, \tag{1a}$$
$$xy = 9\,, \tag{1b}$$

find

$$x\sqrt{x} + y\sqrt{y}\,, \tag{2}$$

where x, y are positive real numbers.

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{3a}$$

$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{3b}$$

$$u_\pm^2 = u_\pm\,, \tag{3c}$$

$$u = u_+ - u_-\,, \tag{3d}$$

$$u_+ u_- = 0\,, \tag{3e}$$

$$u_+ + u_- = 1\,, \tag{3f}$$

$$u u_+ = u_+\,, \tag{3g}$$

$$u u_- = -u_-\,, \tag{3h}$$

$$(u_\pm)^- = u_\mp\,. \tag{3i}$$

You should prove (3c) – (3i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w+z)u_+ + (w-z)u_-\,, \tag{4a}$$

$$wu_+ + zu_- = \tfrac{1}{2}(w+z) + \tfrac{1}{2}(w-z)u\,. \tag{4b}$$

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots. It reminds me of when I was a kid, and other kids would fold a piece of paper in such a way that they could, with two fingers of each hand, open and close the folded paper in two different ways. The practice of this folding is called origami. (Some call the result of that folding the 'Fortune Teller' fold.) But I think of this construction as an analogy: The paper represents a unipode: Open it one way to see the number in the standard basis, and open it the other way to see it in the idempotent basis.

3 The Solution

I tried a lot of first-unipode definition attempts until I settled down on the one I'll show in a moment. Of course, at some point we have to incorporate the given information into the unipodes, and this problem's solution has taught me that the shortest number of steps to the full incorporation should yield the shortest proof/demonstration. And, if you're lucky enough to have a problem that produces a 'pure' unipode, i.e., a unipode whose components are only numbers (or at worst fixed parameters), then whatever typical algebraic operations or basis flipping you do to that unipode, it will return another pure unipode. Let's see how that works!

Let our 'first unipode' be defined as follows:[1]

$$a \equiv \sqrt{x} + \sqrt{y}\,u\,. \tag{5}$$

In order to directly use (1a), we could square the last equation:

$$\begin{aligned} a^2 &= (x+y) + 2\sqrt{xy}\,u\,, & \text{(6a)}\\ &= 19 + 2\sqrt{9}\,u\,, & \text{(6b)}\\ &= 19 + 6\,u\,! & \text{(6c)} \end{aligned}$$

So, the attempt to use (1a) also allowed us to use (1b) as well, and the result is a pure unipode! Now, if we take roots on this unipode or take powers or flip its basis, the result is still a pure unipode. Why is that useful? Because if we do those operations on a unipode that has variables in its components, then those variables may get mixed up between the components. Sometimes that's good, but not always.

Now, let's convert the basis of a^2:

$$a^2 = 25u_+ + 13u_-\,. \tag{7}$$

We're now in the position to extract roots to get the value of a:[2]

$$a = 5u_+ \pm \sqrt{13}\,u_-\,, \tag{8}$$

from which we can take the plus sign on the u_- term.

To get the square roots of x and y, we can convert this last equation back to its standard basis:

$$a = \tfrac{1}{2}(5+\sqrt{13}) + \tfrac{1}{2}(5-\sqrt{13})u\,. \tag{9}$$

On comparing this last equation to (5), gives us

$$\sqrt{x} = \tfrac{1}{2}(5+\sqrt{13})\,, \quad \sqrt{y} = \tfrac{1}{2}(5-\sqrt{13})\,. \tag{10}$$

Therefore,

$$x\sqrt{x} + y\sqrt{y} = (\sqrt{x})^3 + (\sqrt{y})^3 = \left(\frac{5+\sqrt{13}}{2}\right)^3 + \left(\frac{5-\sqrt{13}}{2}\right)^3 = 80\,, \tag{11}$$

where I used Mathematica to do the calculations for me.

[1] A 'first unipode' is our point of entry into the unipodal numbers from either the real numbers of the complex numbers. The choice of this unipode is often neither unique nor obvious to get the most efficient demonstration.

[2] Actually, we get four distinct roots by taking the square root of a unipode. However, the root I finalized on worked according to the constraints on the problem.