

Math Diversion Problem 171

P. Reany

November 21, 2024

Average talent, plus hard work and dedication,
will always beat talent by itself.
— Clinton Anderson

The YouTube video is found at:

Source: https://www.youtube.com/watch?v=uguB0e_8ygs

Title: A Natural Log Problem With Complex Numbers

Presenter: aplusbi

1 The Problem

Given the relation

$$\ln(iz + \sqrt{1 - z^2}) = \frac{i\pi}{3}, \quad (1)$$

find the values of z over the real numbers.

2 The Solution

My idea is to use the following identity in the hyperbolic functions:

$$\ln(x + \sqrt{1 + x^2}) = \sinh^{-1} x, \quad (2)$$

where x can be complex. So, first I have to transform into this form. Let $w = iz$. Then (1) becomes

$$\ln(w + \sqrt{1 + w^2}) = \sinh^{-1} w = \frac{i\pi}{3}. \quad (3)$$

Thus

$$iz = w = \sinh \frac{i\pi}{3}, \quad (4)$$

$$z = \frac{1}{i} \sinh \frac{i\pi}{3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad (5)$$

and, yes, I used another hyperbolic identity.